J Stat Phys (2008) 130: 757–799
DOI 10.1007/s10955-007-9456-0

Self-Similar Behaviour for Noncompactly Supported Solutions of the LSW Model

J.J.L. Velázquez

Received: 4 December 2005 / Accepted: 11 October 2007 / Published online: 20 November 2007

Abstract In this paper a method of approximating the classical LSW model near self-similar solutions for initial data with infinite support is developed. The resulting problem is an integrodifferential equation having two time scales that can be studied using multiple scale methods. The analysis provides a detailed description of the precise manner in which the characteristics "leak" through the critical radius associated to the self-similar solutions. The analysis in this paper makes precise the meaning of the iterated logarithmic asymptotics in the dynamics of the LSW model that were already obtained in the original Lifshitz-Slyozov paper. Examples of noncompactly supported solutions of the LSW model that do not behave in a self-similar manner are also given.

Keywords LSW model · Ostwald ripening · Selfsimilar behaviour · Schwartzian derivative · Integral equations · Multiple scales analysis

1 Introduction

This paper develops a mathematical formalism that makes possible to describe noncompactly supported solutions of the Lifshitz-Slyozov-Wagner model that behave in a self-similar manner for long times.

The Lifshitz-Slyozov-Wagner model (LSW) is the following nonlocal system of differential equations:

$$\frac{\partial f(R,t)}{\partial t}+\frac{\partial}{\partial R}\left(\left(-\frac{1}{R^2}+\frac{\Delta(t)}{R}\right)f(R,t)\right)=0,\quad t>0,\ R>0, \tag{1.1}$$

$$f(R,0)=f_0(R)\geq 0,\quad R>0, \tag{1.2}$$

$$\Delta(t)=\frac{\int_0^\infty f(R,t)dR}{\int_0^\infty Rf(R,t)dR}. \tag{1.3}$$

J.J.L. Velázquez (✉)
Departamento de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense, Madrid 28040, Spain
e-mail: jj_velazquez@mat.ucm.es

These equations are a classical model that describes the last stage of a phase transition process (Ostwald ripening) for which the small volume fraction occupied by the particles is preserved (cf. [6, 16]). Rigorous derivations of this system using homogenization techniques that take as starting point the Mullins-Sekerka free boundary problem have been obtained in different scaling limits in [3, 7, 8, 11].

It is well known that the system (1.1–1.3) has a family of explicit self-similar solutions of the form (cf. [6, 16]):

$$f(R,t) = \frac{1}{t^{4/3}}\Phi_\alpha(\rho), \quad \rho = \frac{R}{t^{1/3}}, \tag{1.4}$$

where $\alpha \in (0, \infty]$. The self-similar solutions (1.4) are compactly supported in the variable ρ. For each given value of the volume fraction $\int_0^\infty \frac{4\pi R^3}{3} f(R,t)dR$ such self-similar solutions are uniquely characterized by means of their asymptotic behaviour near the maximum value of ρ that are given by:

$$\Phi_\beta(\rho) \sim L_\beta(\rho_{\max} - \rho)^\beta \quad \text{as } \rho \to \rho_{\max}^- \tag{1.5}$$

for $-1 < \beta < \infty$ and where $\rho_{\max} := \rho_{\max}(\beta)$, $L_\beta > 0$. On the other hand, for $\beta = \infty$ we have:

$$\Phi_\infty(\rho) \sim L_\infty(\rho_{\max} - \rho)^{-1/3} \exp\left[-\frac{(\frac{3}{2})^{2/3}}{(\rho_{\max} - \rho)}\right] \quad \text{as } \rho \to \rho_{\max}^-, \tag{1.6}$$

where $\rho_{\max} := (\frac{3}{2})^{1/3}$, $L_\infty > 0$. The numerical constants L_β, L_∞ are proportional to the volume fraction occupied by the particles. In (1.5), (1.6) and in the rest of the paper we will use extensively the asymptotic notation whose meaning we recall here:

$$f(x) \sim g(x) \quad \text{as } x \to x_0 \quad \text{iff} \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = 1,$$

$$f(x) \ll g(x) \quad \text{as } x \to x_0 \quad \text{iff} \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0,$$

$$f(x) = o(g(x)) \quad \text{as } x \to x_0 \quad \text{iff} \quad \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0,$$

$$f(x) = O(g(x)) \quad \text{as } x \to x_0 \quad \text{iff} \quad \frac{f(x)}{g(x)} \quad \text{is bounded near } x_0.$$

Global well-posedness of the system (1.1–1.3) for compactly supported initial data has been proved in [10]. On the other hand, it has been rigorously proved in [9] that the initial data $f_0(R)$ must satisfy very stringent assumptions if the corresponding solution $f(R,t)$ behaves as one of the self-similar solutions (1.4) as $t \to \infty$ (see also [4] for a related formal study as well as [1] for an analogous study in a simplified model). Analogous results that include also some necessary conditions on compactly supported initial data that yield the asymptotics (1.6) can be found in [12, 13]. In an informal way the results in [9, 12, 13] can be formulated saying that compactly supported initial data $f_0(R)$ for which the corresponding solutions behave asymptotically as $t \to \infty$ as (1.5), (1.6) must behave near the maximum radius in the same way as these functions. Numerical studies of the asymptotics of the solutions of (1.1–1.3) can be found in [2].

The results of the papers [9, 12, 13] are valid only for compactly supported initial data, but no similar results have been derived in the noncompactly supported case. The goal of this paper is to develop a theory that describes self-similar asymptotics for (1.1–1.3) at the level of formal asymptotics.

In [14] a formal asymptotic expansion for some particular noncompactly supported solutions of the LSW model that decay exponentially as $R \to \infty$ has been obtained. In a strict sense the results in [14] were derived for the Becker-Döring system of equations. Nevertheless the only role played in that paper by the Becker-Döring part of the equation is to transform compactly supported initial data in a noncompactly supported function that decreases exponentially for large radii. Besides this difference, the rest of the analysis in [14] applies without relevant changes to the LSW system. A relevant feature of the analysis in [14] was a detailed analysis of some series that arise in the description of $\Delta(t)$ in (1.1) having the form:

$$\Delta(t) \sim \frac{c_0}{t^{1/3}}\left[1 + c_1\left(\frac{1}{(\log(t))^2} + \frac{1}{(\log(t))^2(\log(\log(t)))^2} + \cdots\right)\right] \tag{1.7}$$

for suitable constants c_0, c_1.

Similar series had been originally obtained in the seminal paper [6]. The analysis of this paper will provide an explanation for the onset of the series (1.7) for noncompactly supported solutions of (1.1–1.3). The idea of using the Becker-Döring system in order to find a selection mechanism for the "correct" self-similar behaviour of the LSW model has been introduced also in [4]. We point out that the series (1.7) arise in the study of the "leaking" of the characteristic curves near the critical radius. It was already noticed in [6] that such "leaking" of characteristics plays a crucial role in the study of solutions behaving in a self-similar manner for long times. The main goal of this paper is to understand such "leaking" of characteristics in a detailed way.

A key point in the study made in the paper(s) [12, 13] was to approximate the dynamics of the compactly supported solutions of the LSW system (1.1–1.3) that remain close to self-similar solutions for long times by means of some integral equations. In this paper we will derive also some integral equations that approximate the LSW dynamics near self-similar solutions for noncompactly supported initial data. There are, however, several differences between the integral equations in [12, 13] and the ones in this paper. Namely, the integral equations that arise for compactly supported initial data are of convolution type and have only one characteristic time scale. On the contrary, the equations derived in this paper for noncompactly supported initial data have two characteristic time scales as $t \to \infty$. Nevertheless, the solutions of these equations can be approximated for long times using multiple scales techniques.

The plan of the paper is the following. In Sect. 2 we recall some basic facts about the self-similar solutions of the LSW model. In Sect. 3 that contains the main results of this paper we study in detail the asymptotic behaviour of the characteristic curves associated to the LSW system yielding self-similar behaviour for the solutions of the problem. Some technical results are collected in several Appendices at the end of the paper.

2 Preliminary Results. Self-Similar Solutions

Some computations will become simpler using as independent variable the volume of the particles v instead of their radius, and replacing the particle density by the cumulative volume distribution. We define $\bar{f}(v,t)$ by means of:

$$v = R^3, \qquad \bar{t} = 3t,$$
$$f(R,t)dR = \bar{f}(v,\bar{t})dv$$

whence $\bar{f}(v,\bar{t})=\frac{1}{3v^{2/3}}f(v^{1/3},t)$. We now introduce the cumulative volume distribution

$$F(v,\bar{t}):=\int_v^\infty \bar{f}(\xi,\bar{t})d\xi. \tag{2.1}$$

Using (1.1–1.3) it follows that $F(v,\bar{t})$ satisfies:

$$\frac{\partial F(v,\bar{t})}{\partial \bar{t}}+(-1+\Delta(\bar{t})v^{1/3})\frac{\partial F(v,\bar{t})}{\partial v}=0,\quad \bar{t}>0,\ v>0, \tag{2.2}$$

$$F(v,0)=F_0(v):=\int_v^\infty \bar{f}_0(\xi)d\xi,\quad v>0, \tag{2.3}$$

$$\Delta(\bar{t})=\frac{3F(0^+,\bar{t})}{\int_0^\infty v^{-2/3}F(v,\bar{t})dv}. \tag{2.4}$$

Due to the positivity of f, $F(\cdot,\bar{t})$ is a decreasing function. The conservation of the total volume of the particles for the LSW model is:

$$\frac{d}{d\bar{t}}\left(\int_0^\infty F(v,\bar{t})dv\right)=0. \tag{2.5}$$

We define a set of self-similar variables by means of:

$$F(v,\bar{t})=\frac{G(W,\tau)}{\bar{t}+1},\qquad W=\frac{2v}{\bar{t}+1},\qquad \tau=\log(\bar{t}+1). \tag{2.6}$$

In this new set of variables (2.2–2.4) becomes:

$$\frac{\partial G}{\partial \tau}+(-2+3\lambda(\tau)W^{1/3}-W)\frac{\partial G}{\partial W}=G, \tag{2.7}$$

$$\lambda(\tau)=\frac{2G(0^+,\tau)}{\int_0^\infty W^{-2/3}G(W,\tau)dW}, \tag{2.8}$$

$$G(W,0)=G_0(W). \tag{2.9}$$

The self-similar solutions of the LSW model are the steady states of the system of equations (2.7–2.9). They are given by the formulas:

$$G_s(W;\lambda_0)=K\exp\left(-\int_0^W\frac{d\xi}{2-3\lambda_0\xi^{1/3}+\xi}\right),\quad 0\le W<W_*,\ G_s(W;\lambda_0)=0$$
$$\text{if } W\ge W_*, \tag{2.10}$$

where $\lambda_0\ge 1$, $K>0$ and W_* is the smallest positive root of the equation:

$$2-3\lambda_0(W_*)^{1/3}+W_*=0. \tag{2.11}$$

Since $W_*\le 1$, for any $\lambda_0\ge 1$ it follows that the solution of the family (2.10) having maximal support is the one corresponding to $\lambda_0=1$. Since that particular solution will be used repeatedly in the following we write it separately by convenience. Moreover, we will normalize the solution assuming that $C=1$ for definiteness:

$$G_s(W)=\exp\left(-\int_0^W\frac{d\xi}{2-3\xi^{1/3}+\xi}\right),\quad 0\le W<1,\ G_s(W)=0 \text{ if } W\ge 1. \tag{2.12}$$

3 On the Conditions for Self-Similar Behaviour for Noncompactly Supported Data

Necessary and sufficient conditions for self-similar behaviour and compactly supported initial data have been studied in [9], as well as in [12, 13]. We will consider from now on the asymptotic of the noncompactly supported solutions of the LSW model that remain close to one of the self-similar solutions during their whole evolution.

3.1 The Only Possible Self-Similar Behaviour Is $\lambda_0 = 1$

It is already implicit in the seminal paper [6] that noncompactly supported solutions of the LSW model (1.1–1.3) that behave asymptotically as a self-similar solution must behave necessarily like the self-similar solution (2.12). We recall here the basic argument that we formulate in the form of a Theorem.

Theorem 1 *Suppose that $G(W,\tau)$ is a solution of (1.1–1.3) with initial data $G_0(W)$ which are nonincreasing, supported in $[0,\infty)$ and satisfying $\int_0^\infty G_0(W)dW < \infty$. Let us suppose also that $\lim_{\tau\to\infty} G(W,\tau) = G_s(W;\lambda_0)$ uniformly in compact sets of W, where $G_s(W;\lambda_0)$ is one of the self-similar solutions in (2.10). Then $\lambda_0 = 1$.*

Proof We argue by contradiction. Let us assume that $\lambda_0 < 1$. The volume conservation (2.5) implies

$$\frac{d}{d\tau}\left(\int_0^\infty G(W,\tau)dW\right) = 0 \tag{3.1}$$

whence:

$$\int_0^\infty G(W,\tau)dW = \int_0^\infty G_0(W)dW < \infty. \tag{3.2}$$

Let us denote as $W(\tau, W_0)$ the solutions of the characteristic equations associated to (2.7) satisfying $W(0, W_0) = W_0$. These functions solve:

$$\frac{dW(\tau,W_0)}{d\tau} = -2 + 3\lambda(\tau)(W(\tau,W_0))^{1/3} - W(\tau,W_0), \tag{3.3}$$

$$W(0,W_0) = W_0. \tag{3.4}$$

Suppose that $G(W,\tau)$ solution of (2.7) approaches to a self-similar solution with $\lambda_0 > 1$. Then $\lambda(\tau) \to \lambda_0$ as $\tau\to\infty$, whence, for τ large enough, the right-hand side of (3.3) vanishes at two values $W_1(\tau)$, $W_2(\tau)$, $0 < W_1(\tau) < W_2(\tau) < \infty$ that approach as $\tau\to\infty$ to the two positive roots of the equation

$$-2 + 3\lambda_0 \bar{W}_i^{1/3} - \bar{W}_i = 0, \quad i = 1, 2,$$

where $0 < \bar{W}_1 < \bar{W}_2 < \infty$.

Using the continuous dependence of the solutions of ordinary differential equations with respect to their initial data, it follows that $\lim_{W_0\to\infty} \frac{W(\tau,W_0)}{W_0} = e^{-\tau}$ uniformly in compact sets $\tau \in [0, L]$, $L > 0$. Choosing L large enough to ensure that $W_1(\tau)$, $W_2(\tau)$ are well defined, it then follows that $W(L, W_0) > W_2(L)$ for W_0 sufficiently large. Since the trajectories associated to the solutions of (3.3) do not intersect in the space (τ, W) it follows that $W(\tau, W_0) > W_2(\tau)$ for $\tau \geq L$. Moreover, the right hand side of (3.3) is negative for $W(\tau, W_0) > W_2(\tau)$. Therefore $W(\cdot, W_0)$ is a decreasing function and

$$\lim_{\tau\to\infty} W(\tau, W_0) = \bar{W}_2 \tag{3.5}$$

for W_0 sufficiently large.

Notice that, since $G_0(\cdot)$ is not compactly supported and nonincreasing, we have $G_0(W_0) > 0$ for any $W_0 \geq 0$. Integrating by characteristics (2.7), (2.9) it follows that $G(W(\tau, W_0), \tau) = G_0(W_0)e^{\tau}$. Using again the monotonicity properties of $G(\cdot)$ as well as (3.5), it follows that:

$$\lim_{\tau\to\infty} \inf\left(\frac{\inf_{W\in[0,\bar{W}_2-\varepsilon_0]} G(W,\tau)}{e^{\tau}}\right) \geq \lim_{\tau\to\infty} \frac{G(W(\tau, W_0), \tau)}{e^{\tau}} > 0 \tag{3.6}$$

for any $\varepsilon_0 > 0$ small, and W_0 sufficiently large. However, (3.6) contradicts (3.2) and the Theorem follows. □

It then follows that the only admissible self-similar behaviour for (2.7) is (2.12). Moreover, the argument above shows that, since G increases exponentially along characteristics, these ones should "leak" across the critical line $W = 1$ at a very precise rate in order to obtain the conservation of volume property (3.1). The rest of the paper is basically a description of the precise way in which this "leaking" of the characteristic curves across the critical line takes place.

3.2 Necessary Conditions for Self-Similar Behaviour

Let us define

$$\beta(\tau) := \lambda(\tau) - 1. \tag{3.7}$$

Since $\int_0^\infty W^{-2/3}\frac{G_s(W)}{G_s(0)}dW = 2$, it follows from (2.8) that:

$$\beta(\tau) = -\frac{\int_0^\infty W^{-2/3}(G(W,\tau) - G(0^+,\tau)G_s(W))dW}{\int_0^\infty W^{-2/3}G(W,\tau)dW} \tag{3.8}$$

and (2.7), (2.9) might be written as:

$$\frac{\partial G}{\partial \tau} + (-2 + 3W^{1/3} - W + 3\beta(\tau)W^{1/3})\frac{\partial G}{\partial W} = G, \tag{3.9}$$

$$G(W,0) = G_0(W). \tag{3.10}$$

We now derive assumptions that must be posed on $G_0(W)$ in order to have the self-similar behaviour (2.12) for the solutions of (3.8–3.10). Using the function $\beta(\tau)$ we can rewrite the characteristic equations for $W(\tau; W_0)$, (3.3), (3.4) as:

$$\frac{dW}{d\tau} = -h(W) + 3\beta(\tau)W^{1/3}, \tag{3.11}$$

$$W(0, W_0) = W_0, \tag{3.12}$$

where:

$$h(W) = 2 - 3W^{1/3} + W.$$

Notice that the function $h(W)$ is a convex function with a minimum at $W=1$ where the following approximation holds:

$$h(W)=\frac{(W-1)^2}{3}+O\big((W-1)^3\big)\quad \text{as } W\to 1. \tag{3.13}$$

For any function $\beta(\tau)$ satisfying $\lim_{\tau\to\infty}\beta(\tau)=0$, we will denote as trajectories the curves contained in the quadrant of the plane $\{(\tau,W):\tau\geq 0,\ W\geq 0\}$ and given by $\{(\tau,W(\tau;W_0)):\tau\geq 0\}$ for each $W_0>0$.

Integrating (3.9) along characteristics we obtain the following identity along the trajectories

$$G(W(\tau;W_0),\tau)=G_0(W_0)e^{\tau}. \tag{3.14}$$

Since $G(W,\tau)$ behaves in a self-similar manner as $\tau\to\infty$ it follows from (3.8) that

$$\beta(\tau)\to 0\quad \text{as } \tau\to\infty. \tag{3.15}$$

The basic idea of the analysis in this Subsection is the following. If $W(\tau;W_0)$ is separated from the critical line $W=1$ the function $h(W)$ is of order one, and due to (3.15) we can approximate (3.11) as:

$$W_\tau=-h(W). \tag{3.16}$$

On the other hand, if $W(\tau;W_0)$ is close to $W=1$ the term $\beta(\tau)$ cannot be ignored in (3.11), but using (3.13) we can approximate (3.11) to the leading order as:

$$W_\tau=-\frac{(W-1)^2}{3}+3\beta(\tau). \tag{3.17}$$

Since for long times the only trajectories left are those starting at $W_0\to\infty$, it is convenient to show that it is possible to use (3.16) to approximate the trajectories having large values of W as well. Notice that for large values of W_0, during the first part of the evolution, for bounded values of τ, $W(\tau;W_0)$ is large and the contribution of the term $3\beta(\tau)W^{1/3}$ in (3.11) is negligible due to the presence of the term $-W$ there. Moreover, if $\tau\to\infty$, $W(\tau;W_0)$ becomes of order one, and due to (3.15) the term $3\beta(\tau)W^{1/3}$ would be negligible too as long as $(-2+3W^{1/3}-W)$ is of order one. Since this function is quadratic in $(W-1)$ as $W\to 1$, it follows that the term $3\beta(\tau)W^{1/3}$ can be neglected also in (3.11) as long as $|W-1|$ remains of order one. Therefore, the combination of these approximations implies that the characteristic trajectories starting at $W=W_0$ with $W_0\gg 1$, might be approximated as $\tau\to\infty$ by means of the solutions of (3.16) as long as $|W-1|$ remains of order one or larger.

The key idea of this paper is to approximate to the leading order the evolution of the characteristics using the explicitly solvable (3.16) for $|W-1|\gtrsim 1$, and (3.17) if $|W-1|$ small. This last equation is not explicitly solvable for an arbitrary function $\beta(\tau)$ that approaches zero as $\tau\to\infty$. Notice that a basic feature of (3.17), that will play a crucial role during the analysis made in this paper, is that the time that the trajectories associated to their solutions spend near the critical line $W=1$ depends on a very sensitive manner in the choice of the function $\beta(\tau)$. Figure 1 shows the typical aspect in the plane $\{(W,\tau)\}$ of the characteristic curves vanishing in a finite time if (3.15) holds.

Let us proceed to analyze the solutions of (3.16). There exists a conserved quantity along the trajectories associated to (3.16) for $W<1$, namely

$$F_{\text{int}}\big(W(\tau,W_0)\big)+\tau, \tag{3.18}$$

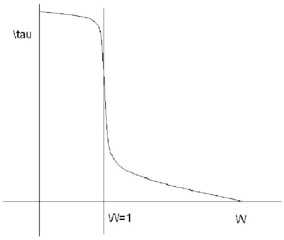

Fig. 1 A typical characteristic curve

where

$$F_{\mathrm{int}}(W) = \int_0^W \frac{d\eta}{2 - 3\eta^{1/3} + \eta}. \tag{3.19}$$

Let us denote as $W_0(\tau; W)$ the inverse function of $W(\tau; W_0)$ for each given τ, or more precisely:

$$W\big(\tau; W_0(\tau; W)\big) = W. \tag{3.20}$$

Taking into account (3.11), (3.18) as well as the fact that $\lim_{\tau\to\infty} \beta(\tau) = 0$, it follows that for any $\alpha \in [0, 1)$

$$W_0(\tau; W) = W_0(\tau + F_{\mathrm{int}}(W) - F_{\mathrm{int}}(\alpha) + \epsilon(\tau, W, \alpha); \alpha), \tag{3.21}$$

where

$$\lim_{\tau\to\infty} \epsilon(\tau, W, \alpha) = 0 \tag{3.22}$$

uniformly on compact sets of $W \in [0, 1)$. Indeed, let us denote as $\bar{\tau}(\alpha; W, \tau)$ the time in which the characteristic curve that reaches the value W at the time τ reaches α. Since the evolution of the characteristics might be approximated by means of (3.16) for large τ, it follows that $\bar{\tau}(\alpha; W, \tau)$ is well defined. Notice that, since we are just considering the evolution of a unique characteristic, we have by definition:

$$W_0(\tau; W) = W_0(\bar{\tau}(\alpha; W, \tau); \alpha).$$

Let us suppose now that the characteristic curves under consideration were given by means of the solutions of the approximate equation (3.16). The conservation of the quantity (3.18) along the trajectories associated to this then implies:

$$\tau + F_{\mathrm{int}}(W) = \bar{\tau}(\alpha; W, \tau) + F_{\mathrm{int}}(\alpha). \tag{3.23}$$

It turns out, however, that the evolution of the characteristic curves is not given by (3.16) but by (3.11). Since $|\tau - \bar{\tau}(\alpha; W, \tau)|$ is bounded for $\alpha, W \in [0, 1)$ and $\lim_{\tau\to\infty} \beta(\tau) = 0$ we can apply classical continuous dependence results for ordinary differential equations to obtain:

$$\tau + F_{\mathrm{int}}(W) + \epsilon(\tau, W, \alpha) = \bar{\tau}(\alpha; W, \tau) + F_{\mathrm{int}}(\alpha),$$

where $\epsilon(\tau, W, \alpha)$ is small if τ is sufficiently large, whence (3.22) follows.

Using (3.14), (3.21) as well as the fact that $G(W, \tau)$ approaches a self-similar solution as $\tau \to \infty$ we obtain

$$\begin{aligned}&G_0(W_0(\tau+F_{\text{int}}(W)-F_{\text{int}}(\alpha)+\epsilon(\tau,W,\alpha);\alpha))\\&\quad=G_0(W_0(\tau;W))=G(W,\tau)e^{-\tau}\\&\quad=Ce^{-\tau}e^{-F_{\text{int}}(W)}[1+o(1)]\end{aligned}\tag{3.24}$$

as $\tau\to\infty$, for some suitable constant C that must be obtained using the volume conservation (3.1).

In particular for each given $\alpha\in[0,1)$, there exists a continuous, strictly increasing function $\omega(\tau)$ such that

$$G_0(\omega(\tau))=Ce^{-F_{\text{int}}(\alpha)}[1+o(1)]e^{-\tau}\tag{3.25}$$

as $\tau\to\infty$, where

$$\omega(\tau)=W_0(\tau;\alpha).\tag{3.26}$$

Condition (3.25) provides a necessary condition that must be satisfied by the initial data G_0 in order to have self-similar behaviour for the solution of (3.8–3.10). However, this condition is less stringent than the ones arising in the compactly supported case (cf. [9, 12]). Indeed, given an arbitrary continuous function $G_0(W)$, which is strictly decreasing, we can find a continuous, strictly decreasing $\omega(\tau)$ such that (3.25) holds by means of:

$$\omega(\tau)=G_0^{-1}\big(Ce^{-F_{\text{int}}(\alpha)}e^{-\tau}[1+o(1)]\big).\tag{3.27}$$

At a first glance, this suggests that any continuous, noncompactly supported, strictly decreasing initial data $G_0(W)$ yields self-similar behaviour as $\tau\to\infty$. Nevertheless, it turns out that this is not the case because, as it is shown in the next Subsection, there exist noncompactly supported initial data $G_0(W)$ for which the long time asymptotics of the solutions of the LSW model is not self-similar. The reason for this is that an initial data, besides satisfying (3.25) for some function $\omega(\tau)$ must also verify that the corresponding function $\beta(\tau)$ that is uniquely defined by means of (3.26) (see Appendix 1), must satisfy

$$\lim_{\tau\to\infty}\beta(\tau)=0,\tag{3.28}$$

since otherwise self-similar behaviour would not be possible.

Moreover, we remark for further reference that the existence of a function $\beta(\tau)$ satisfying (3.28) as well as (3.26) with $\omega(\tau)$ as in (3.27) implies (3.24).

Remark 2 It could seem natural to choose $\alpha=0$ in the previous arguments in order to simplify the computations, but this would introduce some technical complications in the proof of the results in Appendix 1 due to the singular behaviours of the characteristics, solution of (3.11), (3.12) as $W\to 0$. For simplicity, it will be assumed in the rest of the paper that $\alpha=\frac{1}{2}$.

3.3 Nonselfsimilar Behaviour for the LSW Model

If G_0 is allowed to have discontinuities, or equivalently $f_0(R)$ in (1.2) is allowed to contain Dirac masses, it is not hard to find examples of solutions of (3.8–3.10) that do not behave in a self-similar manner as $\tau\to\infty$. Indeed, let us define $G_0(W)$ as follows:

$$G_0(W)=2^{-n},\qquad W\in(n,n+1],\ n=0,1,2,\ldots.\tag{3.29}$$

We begin with the following auxiliary result:

Theorem 3 *The solution of* (3.8–3.10) *with initial data as in* (3.29) *is globally defined for any* $\tau \geq 0$.

Proof Local existence might be obtained adapting the arguments in [10]. In order to prove global existence it is then enough to show that the characteristic trajectories associated to the solutions of (3.11), (3.12) are globally defined. To this end, it is enough to show that $|\beta(\tau)|$ is bounded in any interval $0 \leq \tau \leq T$, for any $T > 0$. In order to show this we notice that, since $\lambda(\tau) = 1 + \beta(\tau) \geq 0$, (3.11), (3.12) yield:

$$W(\tau; W_0) \geq W_0 e^{-\tau}. \tag{3.30}$$

Due to the positivity of $G_0(W)$ for any $W \geq 0$, it follows from (3.30) that $\int_0^\infty W^{-2/3} \times G(W, \tau)dW \geq \delta_T > 0$ for $0 \leq \tau \leq T$. Using (3.8) it then follows that:

$$|\beta(\tau)| \leq C_T \left(G(0^+, \tau) + \int_0^\infty G(W, \tau)dW \right)$$

and due to the volume conservation property (3.1) as well as (3.14) it then follows that $|\beta(\tau)|$ is bounded in any finite interval $0 \leq \tau \leq T$, whence the result follows. □

Theorem 4 *Suppose that* $G(W, \tau)$ *is the solution of* (3.8–3.10) *with initial data* (3.29). *Then* $G(W, \tau)$ *does not behave in a self-similar manner as* $\tau \to \infty$, *or more precisely, the following formula cannot be satisfied*:

$$\lim_{\tau \to \infty} G(W, \tau) = G_s(W; \lambda_0), \quad W \geq 0 \tag{3.31}$$

with $G_s(W; \lambda_0)$ *as in* (2.10) *for any* $K > 0$ *and* $\lambda_0 \geq 1$.

Proof Let us suppose that (3.31) is satisfied for some $K > 0$. Then, due to Theorem 1, $\lambda_0 = 1$. Moreover, Lebesgue's dominated convergence Theorem combined with (3.31) as well as the volume conservation (3.1) implies the following formula for the function $\beta(\tau)$ in (3.8) for any $L > 0$:

$$\begin{aligned}
&\lim_{\tau \to \infty} \sup |\beta(\tau)| \\
&\quad \leq \frac{\lim_{\tau \to \infty} \int_0^L W^{-2/3} |G(W, \tau) - G(0^+, \tau) G_s(W)| dW}{\lim_{\tau \to \infty} \int_0^L W^{-2/3} G(W, \tau) dW} \\
&\qquad + \frac{\frac{1}{L^{2/3}} \sup_\tau \int_L^\infty [G(W, \tau) + G(0^+, \tau) G_s(W)] dW}{\lim_{\tau \to \infty} \int_0^L W^{-2/3} G(W, \tau) dW} \\
&\quad \leq \frac{C}{L^{2/3}},
\end{aligned}$$

where C is independent on L. Choosing L large it then follows that:

$$\lim_{\tau \to \infty} \beta(\tau) = 0.$$

Therefore, the absolute value of the speed of the characteristic curves associated to (3.11) is bounded below for $W \leq \frac{1}{2}$. We can then define a sequence of times τ_n such that:

$$\lim_{n \to \infty} \tau_n = \infty \quad \text{and} \quad W(\tau_n; n) = 0.$$

On the other hand, (3.14) and (3.29) imply that $G(W,\tau)$ is discontinuous at the points $W=W(\tau;n)$ where it satisfies the jump condition:

$$G(W^{-},\tau)-G(W^{+},\tau)=\frac{1}{2}G(W^{-},\tau). \tag{3.32}$$

Using (3.32) we then have:

$$G(0,\tau_n^{-})-G(0,\tau_n^{+})=\frac{1}{2}G(0,\tau_n^{-})$$

but this identity is incompatible with the existence of the limit $\lim_{\tau\to\infty}G(0,\tau)$ unless $C=0$, whence the Theorem follows. □

Remark 5 Notice that for the function G_0 in (3.29) the condition (3.25) fails. Indeed, we have

$$G_0(n^{-})-G_0(n^{+})=\frac{G_0(n^{-})}{2}. \tag{3.33}$$

Condition (3.25) implies

$$\lim_{\tau\to\infty}\frac{G_0(\omega(\tau+a))}{G_0(\omega(\tau))}=e^{-a} \tag{3.34}$$

uniformly on compact sets $a\in[0,\infty)$. However, (3.33), (3.34) are incompatible. Indeed, let us define $\tau_n=\omega^{-1}(n-\frac{1}{2})$. It follows from (3.33) that the function $\frac{G_0(\omega(\tau_n+a))}{G_0(\omega(\tau_n))}$ has a jump discontinuity of at least high $\frac{1}{2}$ at the value $a=\frac{1}{2}$, and this contradicts (3.34).

In the following result we show that in the previous Theorem we do not need to assume that the initial distribution function $f_0(R)$ is a singular measure. On the contrary, it is possible to assume that $f_0\in C^{\infty}(\mathbb{R}^{+})$.

Theorem 6 *There exist functions $G_{0,\varepsilon}\in C^{\infty}(\mathbb{R}^{+})$ which are strictly decreasing and noncompactly supported such that the corresponding solution of the problem* (3.8–3.10) *does not behave in a self-similar manner as $\tau\to\infty$. More precisely* (3.31) *is not satisfied for any $K>0$ and $\lambda_0\geq 1$.*

Proof The basic idea is to construct $G_{0,\varepsilon}(W)$ as a perturbation of the initial data $G_0(W)$ in (3.29). Let us consider a sequence of positive numbers $\{\varepsilon\}:=\{\varepsilon_n>0:n=1,2,\ldots\}$ satisfying

$$\cdots<\varepsilon_{n+1}<\varepsilon_n<\cdots<\varepsilon_1<\frac{1}{4},\quad n=1,2,\ldots$$

and whose values will be precised later. We then define a function $G_{0,\varepsilon}\in C^{\infty}(\mathbb{R}^{+})$ with the following properties:

$$G_{0,\varepsilon}(W)=1,\quad W\in(0,1-\varepsilon_1],$$
$$G_{0,\varepsilon}(W)=2^{-n},\quad W\in[n+\varepsilon_n,n+1-\varepsilon_{n+1}],\ n=1,2,\ldots,$$
$$G_{0,\varepsilon}(W)\text{ strictly decreasing in }\bigcup_{n=1}^{\infty}(n-\varepsilon_n,n+\varepsilon_n).$$

Let us denote as $\beta_\varepsilon(\tau)$ the corresponding function given by (3.8) for the solutions $G_\varepsilon(W,\tau)$ having initial data $G_{0,\varepsilon}(W)$. As a first step we show that the solutions $G_\varepsilon(W,\tau)$ are globally defined in time if the sequence $\{\varepsilon\}$ is small enough in some suitable sense. To this end we derive an estimate on the functions $\beta_\varepsilon(\tau)$ independent on $\{\varepsilon\}$. Let us denote as $W_\varepsilon(\tau; W_0)$ the solution of the characteristic equations (3.11), (3.12) with $\beta(\tau)=\beta_\varepsilon(\tau)$. Differentiating these equations with respect to W_0 we obtain:

$$\frac{d}{d\tau}\left(\frac{\partial W_\varepsilon}{\partial W_0}\right) = -\frac{\partial W_\varepsilon}{\partial W_0} + \lambda_\varepsilon(\tau) W_\varepsilon^{-2/3}\frac{\partial W_\varepsilon}{\partial W_0}, \tag{3.35}$$

$$\frac{\partial W_\varepsilon}{\partial W_0}(0, W_0) = 1, \tag{3.36}$$

where $\lambda_\varepsilon(\tau) = 1+\beta_\varepsilon(\tau)$. Using the positivity of $\lambda_\varepsilon(\tau)$ we obtain:

$$\frac{\partial W_\varepsilon}{\partial W_0} \geq e^{-\tau}. \tag{3.37}$$

We can now derive an upper estimate of $\lambda_\varepsilon(\tau)$ using (2.8). Let us denote as $W_{0,\varepsilon}(\tau; W)$ the inverse of $W_\varepsilon(\tau; W_0)$ for each given value of τ:

$$W_\varepsilon(\tau; W_{0,\varepsilon}(\tau; W)) = W, \quad W \geq 0. \tag{3.38}$$

The starting value of W of the characteristic that vanishes at time τ is then given by $W_{0,\varepsilon}(\tau; 0)$. Our definition of $G_{0,\varepsilon}(W_0)$ implies:

$$G_{0,\varepsilon}(W_0+1) \geq \frac{1}{2} G_{0,\varepsilon}(W_0). \tag{3.39}$$

Using (3.37) we can estimate the value of W at time τ for the trajectory starting at the point $W_{0,\varepsilon}(\tau; 0)+1$ at time $\tau=0$. We have:

$$W_\varepsilon(\tau; W_{0,\varepsilon}(\tau; 0)+1) \geq e^{-\tau}.$$

Then, using also (3.39):

$$\begin{aligned}
\int_0^\infty W^{-2/3} G(W,\tau)\,dW &\geq \int_0^{W_\varepsilon(\tau; W_{0,\varepsilon}(\tau;0)+1)} W^{-2/3} G(W,\tau)\,dW \\
&\geq \int_0^{W_\varepsilon(\tau; W_{0,\varepsilon}(\tau;0)+1)} W^{-2/3} G(W_\varepsilon(\tau; W_{0,\varepsilon}(\tau; 0)+1),\tau)\,dW \\
&= e^\tau G_{0,\varepsilon}(W_{0,\varepsilon}(\tau; 0)+1) \int_0^{W_\varepsilon(\tau; W_{0,\varepsilon}(\tau;0)+1)} W^{-2/3}\,dW \\
&\geq \frac{3e^\tau G_{0,\varepsilon}(W_{0,\varepsilon}(\tau; 0))}{2} (W_\varepsilon(\tau; W_{0,\varepsilon}(\tau; 0)+1))^{1/3} \\
&\geq \frac{3}{2} G(0^+,\tau) e^{-\frac{\tau}{3}}.
\end{aligned}$$

Therefore (2.8) implies the following estimate:

$$0 \leq \lambda_\varepsilon(\tau) \leq \frac{4}{3} e^{\frac{\tau}{3}}. \tag{3.40}$$

Estimate (3.40) yields global existence of the solutions $G_\varepsilon(W,\tau)$ having as initial data $G_{0,\varepsilon}$.

As a next step we need to derive an upper estimate which is uniform in ε, for the time that a characteristic starting at W_0 remains in the region $W>0$. For each $W_0>0$, let us define:

$$T_\varepsilon^*(W_0):=\inf\{\tau>0: W_\varepsilon(\tau;W_0)=0\}. \tag{3.41}$$

Using the volume conservation (3.1) we can derive some auxiliary estimates for $W_\varepsilon(\tau;W_0)$. Indeed, notice that for any $\tau<T_\varepsilon^*(W_0)$ we have:

$$C_\varepsilon:=\int_0^\infty G_\varepsilon(W,\tau)dW\geq\int_0^{W_\varepsilon(\tau;W_0)}G_\varepsilon(W,\tau)dW\geq G_\varepsilon(W_\varepsilon(\tau;W_0),\tau)W_\varepsilon(\tau;W_0),$$

where $C_\varepsilon=\int_0^\infty G_{0,\varepsilon}(W)dW$ is uniformly bounded above and below for the class of values $\{\varepsilon\}$ under consideration. Using then (3.14) we obtain:

$$C\geq G_{0,\varepsilon}(W_0)e^\tau W_\varepsilon(\tau;W_0),$$

where C is independent on $\{\varepsilon\}$. Then, since $G_{0,\varepsilon}(W_0)\geq e^{-bW_0}$ for some $b>0$ independent on $\{\varepsilon\}$ we obtain:

$$W_\varepsilon(\tau;W_0)\leq Ce^{bW_0}e^{-\tau}. \tag{3.42}$$

Notice that (3.42) implies that for $\tau\geq\tau_0(W_0):=bW_0+\log(C)$ we have:

$$W_\varepsilon(\tau;W_0)\leq 1. \tag{3.43}$$

We can now derive the desired upper estimate for $T_\varepsilon^*(W_0)$. Using again (3.1) and (3.14) we obtain:

$$C_\varepsilon:=\int_0^\infty G_\varepsilon(W,\tau)dW=e^\tau\int_{W_{0,\varepsilon}(\tau;0)}^\infty G_{0,\varepsilon}(W_0)\frac{\partial W_\varepsilon}{\partial W_0}(\tau;W_0)dW_0. \tag{3.44}$$

Suppose that $T_\varepsilon^*(W_0)>\tau_0(W_0)$, since otherwise the desired uniform upper estimate would be already derived. Integrating (3.35), (3.36) we derive the following inequality for $\tau_0(W_0)\leq\tau\leq T_\varepsilon^*(W_0)$:

$$\frac{\partial W_\varepsilon}{\partial W_0}(\tau;W_0)=e^{-\tau}e^{\int_0^\tau\lambda_\varepsilon(s)W_\varepsilon^{-2/3}(s;W_0)ds}\geq e^{-\tau}e^{\int_{\tau_0(W_0)}^\tau\lambda_\varepsilon(s)W_\varepsilon^{-2/3}(s;W_0)ds}.$$

Using (3.43) it follows that:

$$\frac{\partial W_\varepsilon}{\partial W_0}(\tau;W_0)\geq e^{-\tau}e^{\int_{\tau_0(W_0)}^\tau\lambda_\varepsilon(s)ds}$$

and plugging this inequality into (3.44), combined with $G_{0,\varepsilon}(W_0)\geq e^{-bW_0}$ we obtain:

$$e^{\int_{\tau_0(W_0)}^\tau\lambda_\varepsilon(s)ds}\leq Ce^{bW_0}$$

whence:

$$\int_{\tau_0(W_0)}^\tau\lambda_\varepsilon(s)ds\leq\tau_0(W_0)=bW_0+C. \tag{3.45}$$

Due to (3.43), the equation that defines the characteristic curves (3.11) can then estimated for $\tau \geq \tau_0(W_0)$ as:

$$W_{\varepsilon,\tau} \leq -2 - W + 3\lambda_\varepsilon(\tau) \leq -2 + 3\lambda_\varepsilon(\tau).$$

Then, using again (3.43) as well as (3.45):

$$W_\varepsilon(\tau; W_0) \leq 1 - 2(\tau - \tau_0(W_0)) + 3\tau_0(W_0)$$

and this inequality finally yields:

$$T_\varepsilon^*(W_0) \leq \tau_0(W_0) + \frac{1 + 3\tau_0(W_0)}{2} \leq C(1 + W_0) \tag{3.46}$$

for some $C > 0$. This is the sought-for uniform estimate for the lifetime of each characteristic curve.

In order to conclude the proof of the Theorem we just need to show that choosing $\{\varepsilon\}$ small in some suitable sense, the corresponding solution of (3.8–3.10) $G_\varepsilon(W, \tau)$ cannot approach to a self-similar solution.

Notice that using a Gronwall estimate in (3.11), (3.12) combined with the uniform estimate (3.40) we obtain, for $0 \leq \tau \leq \tau^*$:

$$|W_\varepsilon(\tau; W_{0,1}) - W_\varepsilon(\tau; W_{0,2})| \leq h(\tau^*)|W_{0,1} - W_{0,2}| \tag{3.47}$$

for any $W_{0,1},\ W_{0,2} \in (0, \infty)$ as long as both trajectories are defined. From now on we denote as $h(\cdot)$ a generic increasing function, bounded in any compact set of $[0, \infty)$.

Let us assume that $W_{0,1} = n - \varepsilon_n,\ W_{0,2} = n + \varepsilon_n$. Then (3.47) becomes:

$$|W_\varepsilon(\tau; n - \varepsilon_n) - W_\varepsilon(\tau; n + \varepsilon_n)| \leq 2h(\tau^*)\varepsilon_n. \tag{3.48}$$

Notice that due to (3.46) the trajectories $W_\varepsilon(\tau; n - \varepsilon_n)$, $W_\varepsilon(\tau; n + \varepsilon_n)$ vanish at some times $T_\varepsilon^*(n - \varepsilon_n) \leq Cn$, $T_\varepsilon^*(n - \varepsilon_n) \leq Cn$.

Using (3.11), (3.12) and (3.40) it follows that there exists $\delta_n > 0$ such that, for any $0 \leq \tau_0 \leq Cn$, two trajectories that at time $\tau = \tau_0$ are contained in the interval $W \in (0, \delta_n)$ disappear in a time smaller than δ_n and the vanishing times are separated less than an amount $C\delta_n$. Indeed, notice that to show this result it is enough to choose δ_n satisfying $\lambda(\tau)(\delta_n)^{1/3} \leq e^{Cn}(\delta_n)^{1/3}$ sufficiently small.

Suppose that we then select ε_n such that $2h(Cn)\varepsilon_n \leq \min(\frac{\delta_n}{2}, \frac{1}{n})$. Then, due to (3.48) we obtain that the trajectories starting at the points $W_0 = n - \varepsilon_n$ and $W_0 = n + \varepsilon_n$ disappear respectively at two times $T_\varepsilon^*(n - \varepsilon_n)$ and $T_\varepsilon^*(n + \varepsilon_n)$ satisfying:

$$0 \leq T_\varepsilon^*(n + \varepsilon_n) - T_\varepsilon^*(n - \varepsilon_n) \leq \frac{C}{n}. \tag{3.49}$$

Since

$$G(0^+, T_\varepsilon^*(n + \varepsilon_n)) = e^{T_\varepsilon^*(n+\varepsilon_n)} G_0(n + \varepsilon_n),$$
$$G(0^+, T_\varepsilon^*(n - \varepsilon_n)) = e^{T_\varepsilon^*(n-\varepsilon_n)} G_0(n - \varepsilon_n)$$

it then follows from the fact that $G_0(n - \varepsilon_n) - G_0(n + \varepsilon_n) \geq \frac{G_0(n-\varepsilon_n)}{2}$ and from (3.49) that $G(0^+\tau)$ cannot converge to a limit as $\tau \to \infty$ whence the Theorem follows. $\square$

3.4 Approximating the Solutions of the LSW Model that Are Close to Self-Similar Solutions for Long Times

3.4.1 Sketch of the Main Ideas

Our goal is to approximate the dynamics of the LSW model for noncompactly supported solutions that remain close to self-similar solutions in some suitable sense. The most natural idea would be try to linearize the LSW model near the self-similar solution given in (2.12). Unfortunately such type of linearizations are not so simple in the LSW system, even for compactly supported solutions. In the case of noncompactly supported solutions the structure of the characteristic curves given by (3.11) in the plane $\{(\tau, W)\}$ is very different for the explicit self-similar solution (2.12) and for a noncompactly solution approaching to (2.12) as $\tau \to \infty$. Indeed, in the first case all the characteristic curves where $G > 0$ are contained in the half-strip $\{(\tau, W) : \tau \geq 0,\ 0 \leq W \leq 1\}$. On the contrary, in the case of noncompactly supported solutions the only characteristic curves that are needed to describe the long time asymptotics of the solutions as $\tau \to \infty$ are those beginning at $W = W_0$ for $\tau = 0$, with $W_0 \to \infty$. The dynamics of these characteristic curves that have been described in Sect. 3.2 can be separated into three different stages. In the first stage the characteristic curves might be approximated using (3.16) and W decreases from W_0 to some value close to one but larger. During the second stage, the characteristics can be approximated using (3.17) and W remains close to one. Finally in the last stage W is smaller than one and $|W - 1|$ becomes of order one. During this third stage the dynamics of the characteristics can be approximated again using (3.16).

The basic idea of this paper consists in using the description of the characteristics sketched above in order to derive some problems, simpler than the whole LSW system, but easier to analyze. In the rest of this paper we will be concerned with the study of three problems that we state by decreasing order of difficulty:

(i) The whole LSW model near the selfsimilar solution (2.12) (cf. (3.8), (3.9), (3.10)).
(ii) The transition problem (cf. (3.53), (3.54) below).
(iii) The approximate transition problem (cf. (3.75), (3.76) below).

A rather peculiar feature of the LSW model is the fact that the function $\beta(\tau)$ that has a crucial effect in the evolution of the characteristic curves depends on the solution of the problem itself by means of the nonlocal condition (3.8). On the contrary in the transition problem mentioned above, $\beta(\tau)$ will not be chosen by means of the nonlocal condition, but instead it will be chosen as the function that produces such a transformation in the characteristic curves given by (3.11), (3.12) that the function G obtained by means of (3.14) and that it will be denoted as $G_{\text{trans}}(W, \tau)$ from now on, satisfies:

$$G_{\text{trans}}\left(\frac{1}{2}, \tau\right) = G_s\left(\frac{1}{2}\right)(1 + o(1)), \quad \text{as } \tau \to \infty, \tag{3.50}$$

where $G_s(W)$ is as in (2.12).

The transition problem described above is not a standard initial value problem for ordinary differential equations. Actually, this problem has more analogies, from the mathematical point of view, with an inverse scattering problem. In this case the data is $G_0(W)$ (or equivalently, due to (3.27) $\omega(\tau)$), and the function to be obtained is $\beta(\tau)$.

At a first glance the original LSW model (i) and the transition problem (ii) look very unrelated, but it turns out that this is not the case. It is not hard to see, from the dynamics of the characteristic curves and using (3.27) that, if

$$\lim_{\tau\to\infty}\beta(\tau)=0 \tag{3.51}$$

the function $G_{\text{trans}}(W,\tau)$ satisfies:

$$G_{\text{trans}}(W,\tau)\sim G_s(W)\quad \text{as } \tau\to\infty \tag{3.52}$$

uniformly on compact sets of W if (3.27). Since the right hand side of (3.52) is a solution of the whole LSW model it follows that $G_{\text{trans}}(W,\tau)$ provides an approximate solution of the LSW model as $\tau\to\infty$. Therefore, if (3.51) is satisfied the solution of the transition problem (ii) provides an approximation of the whole LSW dynamics.

Let us explain now the role played by the approximate transition problem (iii) in this paper. It turns out that it is easy to prove local solvability of the transition problem (i) (cf. Appendix 1), but it is not so simple to obtain information about the long time asymptotics of its solutions. However, it turns out that, under the assumption (3.51), it is possible to approximate for long times the transition problem (ii) using the approximate transition problem (iii) that is explicitly solvable. The key idea in the derivation of the approximate transition problem is to replace the evolution of the characteristic curves (3.11) by the simpler equation (3.17). On the other hand, the characteristic curves are approximated for the values of τ where $|W-1|\gtrsim 1$ using the approximate equation (3.16) that can be solved explicitly. Using these approximations the condition (3.50) becomes a condition in the time that a characteristic curve, that "arrives" at a time τ_1 to the critical line, remains "trapped" near such a critical line (cf. (3.71), (3.74)). Precise definitions of the "arrival time" and "trapping time" are given later.

As it was mentioned above, it turns out that the approximate transition problem (iii) can be explicitly solved (cf. Sect. 3.4.3). The reason that underlies the solvability of the approximate transition problem is the fact that the Riccati equation (3.17) can be transformed into a second order linear equation. The condition that prescribes the trapping time of the trajectories can be reformulated as a condition on the distance between consecutive zeroes of the solutions of the second order linear equation. The problem then becomes the one of reconstructing the "potential" $\beta(\tau)$ given the distance between zeros of the solutions, and this can be made using general properties of second order linear equations. In this paper we have not followed such approach, but the solution given in this paper of the approximate transition problem (iii) is convenient in order to study the long time asymptotics of the transition problem (ii) in a perturbative manner.

We can now describe the general strategy that will be followed in this paper to study self-similar asymptotics of the solutions of the LSW model (i). We will first formulate in a precise manner the transition problem (ii) and the approximate transition problem (iii) (cf. Sect. 3.4.2). The explicit solution of the approximate transition problem (iii) will be then given in Sect. 3.4.3. Using this explicit solution we will be able to develop a formalism that describes the long time asymptotics of the solutions of the transition problem (ii) in Sect. 3.4.4 if (3.51) holds. Finally, since the transition problem (ii) approximates the whole LSW model (i) if (3.51) is satisfied, as it was explained above, it is reasonable to "linearize" the LSW model near the transition problem (ii). This is made in Sect. 3.5.

It is natural to ask if it would not be simpler to approximate directly the LSW model (i) using the approximate transition problem (iii). The main difficulty that we have found with that approach is that the "trapping time" near the critical line is rather sensitive to the nonlinear terms $O((W-1)^3)$ that have been ignored in the approximation (3.17). As a consequence the solutions of the approximate transition problem (iii) do not satisfy in an approximate manner the volume conservation property (3.1). In particular this makes the approximate transition problem (iii) less convenient in order to linearize the whole LSW model (i). For this reason, we have found more it convenient to approximate the LSW dynamics using the transition problem (ii) as an intermediate step that allows to control more easily the effect of the above mentioned terms of order $O((W-1)^3)$.

3.4.2 The Transition Problem and the Approximate Transition Problem

3.4.2.1 The Transition Problem As indicated above, we first define a problem that approximates asymptotically the LSW dynamics near self-similar solutions. In a more precise manner, notice that the evolution of each characteristic curve depends, by means of the function $\beta(\tau)$ in (3.8) on the values of G in all the characteristic curves at a given time τ. We have seen in the previous Subsection that for solutions behaving in a self-similar manner, if $\omega(\tau)$ is given by (3.26), (3.25) is satisfied. In Appendix 1 it is shown that the function $\beta(\tau)$ is uniquely prescribed by means of the function $W_0(\tau;\alpha)$ for any fixed value of $\alpha \in (0,1)$. The term $o(1)$ in (3.25) does not modify the self-similar behaviour of $G(W,\tau)$. Therefore, the LSW model might be approximated by the following auxiliary problem:

To find $\beta(\tau)$ such that the function $W(\tau; W_0)$ solution of:

$$W_\tau = -2 + 3W^{1/3} - W + 3\beta(\tau)W^{1/3}, \tag{3.53}$$

$$W(0; W_0) = W_0 \tag{3.54}$$

satisfies:

$$W(\tau; \omega(\tau)) = \alpha = \frac{1}{2}, \tag{3.55}$$

where:

$$\omega(\tau) = G_0^{-1}\left(Ce^{-F_{\mathrm{int}}(\frac{1}{2})}e^{-\tau}\right) \tag{3.56}$$

with:

$$C = \frac{\int_0^\infty G_0(W)dW}{\int_0^\infty e^{-F_{\mathrm{int}}(W)}dW}. \tag{3.57}$$

The main advantage of the problem (3.53–3.57), that will be denoted from now on as "transition problem", is that the integral term (3.8) disappears. Moreover, problem (3.53–3.57) can be approximated, as it will be seen below, by an explicitly solvable problem as $\tau \to \infty$. Of course, it is not possible to ensure that for functions $G_{\mathrm{trans}}(W,\tau)$ obtained by means of the evolution by characteristics (3.11), (3.12) and (3.14), and $\beta(\tau)$ chosen solving the transition problem, the resulting function $\beta(\tau)$ will be given by the integral formula (3.8) and in general this will not happen. On the other hand, since, as we will see, $\beta(\tau) \to 0$ as $\tau \to \infty$, it turns out that the function $G_{\mathrm{trans}}(W,\tau)$ so obtained will behave like the self-similar solution $G_s(W)$ given in (2.12) (up to a multiplicative constant). Since the formula for $\beta(\tau)$ in (3.8) is just a reformulation of the volume conservation property (3.1) and the volume conservation property is satisfied for the self-similar solutions (2.12) we can expect $G_{\mathrm{trans}}(W,\tau)$ to be an approximation of the solution of the LSW problem, and in particular to satisfy (3.8) up to some approximation.

3.4.2.2 The Approximate Transition Problem We now proceed to derive an "approximate transition problem", that in some sense is close to the transition problem (3.53–3.57) as $\tau \to \infty$, and it has the advantage of being explicitly solvable. To this end we need to describe the behaviour of the characteristic curves associated to (3.53–3.57) for long times. The characteristics arriving at $W \approx 1$ at a time $\tau \gg 1$ start their motion at some $W_0 \gg 1$. As long as $W - 1$ is of order one the solutions of (3.53) can be approximated by means of the solutions of (3.16) that are given by

$$W(\tau; W_0) = w_{\text{ext}}(\tau - \log(W_0) + F_{\text{ext}}(W_0)), \tag{3.58}$$

where:

$$-\log(w_{\text{ext}}(s)) + F_{\text{ext}}(w_{\text{ext}}(s)) = s, \tag{3.59}$$

$$F_{\text{ext}}(W) := \int_W^\infty \left[\frac{1}{2 - 3\eta^{1/3} + \eta} - \frac{1}{\eta}\right] d\eta = \int_W^\infty \frac{(3\eta^{1/3} - 2)}{(2 - 3\eta^{1/3} + \eta)\eta} d\eta. \tag{3.60}$$

Notice that:

$$F_{\text{ext}}(W) = \frac{3}{(W-1)} - \frac{5}{3}\log(W-1) + O(1) \quad \text{as } W \to 1^+. \tag{3.61}$$

After reaching the region where W becomes of order one the characteristics remain trapped in the region $W \approx 1$ during a very long time, due to the smallness of $\beta(\tau)$, before crossing to the region $W < 1$. This stage is the crucial part of the evolution of the characteristics. In this part of the dynamics the equation for the characteristics can be approximated by means of (3.17). Using (3.59–3.61) we obtain the following asymptotics:

$$w_{\text{ext}}(s) \sim 1 + \frac{3}{s} - \frac{5\log(\frac{3}{s})}{s^2} + O\left(\frac{1}{s^2}\right) \quad \text{as } s \to \infty. \tag{3.62}$$

It then follows from (3.58) that in the "transition region" where $W \approx 1$, the characteristic $W(\tau; W_0)$ matches in the region where $(\tau - \log(W_0) + F_{\text{ext}}(W_0)) \gg 1$, $(W-1)^2 \gg |\beta(\tau)|$ with the unique solution of (3.17) that satisfies

$$W = +\infty \quad \text{at } \tau = \log(W_0) - F_{\text{ext}}(W_0). \tag{3.63}$$

On the other hand, let us suppose that the characteristic curve $W(\tau; W_0)$ satisfies $W(\bar{\tau}; W_0) = \alpha$, where $\alpha = \frac{1}{2}$ and $\bar{\tau}$ is a large number. Using the approximation (3.16) in the region where $(1 - W)$ is of order one it would follow that

$$W(\tau; W_0) = w_{\text{int}}(\tau - \bar{\tau} - F_{\text{int}}(\alpha)), \tag{3.64}$$

where F_{int} is as in (3.19) and w_{int} is defined by means of:

$$F_{\text{int}}(w_{\text{int}}(s)) = -s. \tag{3.65}$$

Notice that

$$F_{\text{int}}(W) \sim \frac{3}{(1-W)} + \frac{5}{3}\log(1-W) + O(1) \quad \text{as } W \to 1^-. \tag{3.66}$$

Combining (3.65) and (3.67) we obtain

$$w_{\text{int}}(s) \sim 1 + \frac{3}{s} - \frac{5\log(-\frac{3}{s})}{s^2} + O\left(\frac{1}{s^2}\right) \quad \text{as } s \to -\infty. \tag{3.67}$$

Let us denote as $\hat{W}$ the unique solution of (3.17) satisfying:

$$\hat{W} = -\infty \quad \text{at } \tau = \bar{\tau} + F_{\text{int}}(\bar{W}). \tag{3.68}$$

The formulas (3.64) and (3.67) imply that $W(\tau; W_0)$ matches with $\hat{W}$ for the values of τ satisfying:

$$(\tau - \bar{\tau} - F_{\text{int}}(\alpha)) \gg 1, \qquad (\hat{W} - 1)^2 \gg |\beta(\tau)|. \tag{3.69}$$

Indeed, we are interested in the study of (3.17) with $|\beta(\tau)| \to 0$ as $\tau \to \infty$. For such a solutions and, as long as $(W-1)^2 \gg |\beta(\tau)|$ we can approximate the solutions of (3.17) by means of the solutions of $\hat{W}_\tau = -\frac{(\hat{W}-1)^2}{3}$. The solutions of this equation satisfying (3.68) have the form:

$$\hat{W} = 1 + \frac{3}{(\tau - \bar{\tau} - F_{\text{int}}(\alpha))}$$

and due to (3.64) as well as the asymptotics (3.67), $\hat{W}$ matches with $W(\tau; W_0)$ if (3.69) is satisfied. Notice that, at a first glance there seems to be something paradoxical in the fact that $|\hat{W}|$ could become unbounded near the points $\tau = \log(W_0) - F_{\text{ext}}(W_0)$ and $\tau = \bar{\tau} + F_{\text{int}}(\alpha)$, since the functions $W(\tau; W_0)$ remains always bounded. However, there is not such a paradox, because the function W is approximated by means of the solution of (3.17) satisfying (3.63) only in those regions where $|\hat{W} - 1| \ll 1$. The use of a boundary condition like (3.68) where $\hat{W}$ takes unbounded values is just a convenient normalization condition that will simplify the analysis of the dynamics of the characteristics (3.11), (3.12) near the critical line $W = 1$.

In order to simplify the notation we define $X := \hat{W} - 1$. Combining (3.17), (3.63), it follows that X satisfies for each $\bar{\tau}$

$$X_\tau = -\frac{X^2}{3} + 3\beta(\tau), \quad S(\bar{\tau}) < \tau < \bar{\tau}, \tag{3.70}$$

$$X(S(\bar{\tau})) = +\infty, \qquad X(\bar{\tau}) = -\infty, \tag{3.71}$$

where the function $S(\bar{\tau})$ has been defined by means of the equation

$$S(\bar{\tau} + F_{\text{int}}(\alpha)) = \log(W_0) - F_{\text{ext}}(W_0). \tag{3.72}$$

In particular, choosing $\alpha = \frac{1}{2}$, and using (3.55) we obtain that

$$S\left(\bar{\tau} + F_{\text{int}}\left(\frac{1}{2}\right)\right) = \log(\omega(\bar{\tau})) - F_{\text{ext}}(\omega(\bar{\tau})), \tag{3.73}$$

where $\omega(\bar{\tau})$ is as in (3.56), whence

$$S(\tau) = \log\left(\omega\left(\tau - F_{\text{int}}\left(\frac{1}{2}\right)\right)\right) - F_{\text{ext}}\left(\omega\left(\tau - F_{\text{int}}\left(\frac{1}{2}\right)\right)\right). \tag{3.74}$$

Therefore, as $\tau \to \infty$, the problem (3.53–3.57) might be approximated by means of (3.70), (3.71) where the function $S(\cdot)$ is defined by means of (3.56), (3.74). This "approximated transition problem" is the main ingredient in order to study the asymptotics of both the transition problem (3.53–3.57) and the whole LSW model as $\tau \to \infty$.

3.4.3 On the Explicit Solution of the Approximate Transition Problem

The approximate transition problem (3.56), (3.70), (3.71), (3.74) can be explicitly solved. We will assume by convenience that the function $S(\tau)$ defined by means of (3.74) has three derivatives, something that due to (3.56), is equivalent to assuming that G_0 is differentiable enough.

We rewrite here the approximate transition problem for convenience. The problem is, the following:

Problem 7 (Approximate transition problem) Given $T_0 > 0$ and a strictly monotonic increasing function $S(\cdot)$ defined in $[T_0, \infty)$ satisfying $0 < S(\tau) < \tau$ for any $\tau \in [T_0, \infty)$ and $\lim_{\tau\to\infty} S(\tau) = \infty$, to find a function $\bar{\beta}(\tau)$ defined for $\tau \in [S(T_0), \infty)$ such that for any $\bar{\tau} > T_0$:

$$X_\tau = -\frac{1}{3}X^2 + 3\bar{\beta}(\tau), \quad S(\bar{\tau}) < \tau < \bar{\tau}, \tag{3.75}$$

$$X\big((S(\bar{\tau}))^+\big) = +\infty, \qquad X\big((\bar{\tau})^-\big) = -\infty. \tag{3.76}$$

3.4.3.1 A Particular Solution of the Approximate Transition Problem There are several different ways of solving the problem (3.75), (3.76). The method used in this Section is convenient in order to treat perturbatively the whole transition problem (3.53–3.57).

Let us define for $\zeta \geq \zeta_0$ a function $f(\zeta) \geq S(T_0)$ which is strictly monotonically increasing and satisfying:

$$S(f(\zeta+1)) = f(\zeta), \tag{3.77}$$

$$f(\zeta_0) = S(T_0), \tag{3.78}$$

where $S(\cdot)$ is the function in (3.76). Notice that (3.77), (3.78) do not define the function $f(\zeta)$ uniquely. However, if we prescribe an arbitrary function $f(\zeta)$ in $[\zeta_0, \zeta_0+1)$ satisfying (3.78) we obtain a unique f defined in $[\zeta_0, \infty)$ iterating the formula:

$$f(\zeta+1) := S^{-1}(f(\zeta)), \tag{3.79}$$

where $S^{-1}(\cdot)$ is the inverse function of $S(\cdot)$ that is well defined due to our assumptions on this last function.

In further computations we will need f to be three times differentiable. The function f defined by means of (3.79) has this regularity if the function $f(\zeta)$ defined in $[\zeta_0, \zeta_0+1)$ that will be assumed to be monotonically increasing in this interval belongs to $C^3[\zeta_0, \zeta_0+1]$ and satisfies the following compatibility conditions:

$$f(\zeta_0) = f(\zeta_0+1), \tag{3.80}$$

$$f'(\zeta_0) = S'(f(\zeta_0+1))f'(\zeta_0+1), \tag{3.81}$$

$$f''(\zeta_0) = S''(f(\zeta_0+1))(f'(\zeta_0+1))^2 + S'(f(\zeta_0+1))f''(\zeta_0+1), \tag{3.82}$$

$$f'''(\zeta_0) = S'''(f(\zeta_0+1))(f'(\zeta_0+1))^3, \\ + 3S''(f(\zeta_0+1))f'(\zeta_0+1)f''(\zeta_0+1) + S'(f(\zeta_0+1))f'''(\zeta_0+1). \tag{3.83}$$

Under these assumptions the monotonically increasing function f defined by means of (3.79) belongs to $C^3[\zeta_0,\infty)$ and it satisfies:

$$\lim_{\zeta\to\infty} f(\zeta) = \infty.$$

Theorem 8 *Given $f \in C^3[\zeta_0, \zeta_0+1]$ satisfying the compatibility conditions* (3.80–3.83) *we can obtain $f \in C^3[\zeta_0,\infty)$ by means of* (3.79). *Let us denote as $\{f,\zeta\}$ the Schwartzian derivative of f and given by* (*cf.* [5]):

$$\{f,\zeta\} := \frac{f'''(\zeta)}{f'(\zeta)} - \frac{3}{2}\left(\frac{f''(\zeta)}{f'(\zeta)}\right)^2. \tag{3.84}$$

Then, the function $\bar{\beta}(\tau)$ defined in parametric form by means of:

$$\bar{\beta}(\tau) = \frac{1}{(f'(\zeta))^2}\left(\frac{1}{2}\{f,\zeta\} - \pi^2\right), \\ \tau = f(\zeta) \tag{3.85}$$

provides a particular solution of the approximate transition problem (3.75–3.76).

Proof Given $f(\cdot)$ as in the statement of the Theorem we define a new function $\chi(Y,\zeta)$ as:

$$\chi(Y,\zeta) := \frac{3Y}{f'(\zeta)} + \frac{3f''(\zeta)}{2(f'(\zeta))^2}. \tag{3.86}$$

Let us introduce the change of variables:

$$\tau = f(\zeta), \tag{3.87}$$

$$X = \chi(Y,\zeta). \tag{3.88}$$

Suppose that we take $\bar{\beta}(\tau)$ in (3.75), (3.76) as the one defined by means of the parametric formula (3.85). Using the change of variables (3.87), (3.88) it turns out that the approximate transition problem (3.75–3.76) becomes, after some lengthy computations:

$$\frac{dY}{d\zeta} + Y^2 + \pi^2 = 0, \quad \bar{\zeta} < \zeta < \bar{\zeta}+1, \tag{3.89}$$

$$Y((\bar{\zeta})^+) = +\infty, \tag{3.90}$$

$$Y((\bar{\zeta}+1)^-) = -\infty, \tag{3.91}$$

where $\bar{\tau} = f(\bar{\zeta})$.

Notice that a solution of (3.89–3.91), assuming that it exists, would provide an explicit solution of (3.75), (3.76) $\bar{\beta}$ given by (3.85) and the corresponding $X(\tau)$ given in parametric form by means of (3.87), (3.88).

Using the fact that (3.89) can be integrated explicitly we can obtain the solution of the problem (3.89–3.91). Indeed, the unique solution of (3.89), (3.90) is given by:

$$Y = Y(\zeta, \zeta_0) = \pi \cot(\pi(\zeta - \bar{\zeta})) \tag{3.92}$$

and it can be immediately checked that $Y(\zeta, \zeta_0)$ satisfies (3.91), whence the Theorem follows. □

3.4.3.2 On the General Solution of the Approximate Transition Problem In this Subsection we obtain the general solution of (3.75–3.76). The main result is the following:

Theorem 9 *Let us denote as $\bar{\beta}(\tau)$ an arbitrary solution of* (3.75–3.76) *given by* (3.85) *for some fixed function $f(\zeta)$. Then, an arbitrary solution $\beta(\tau)$ of* (3.75–3.76) *can be written in the form*:

$$\beta(\tau) = \bar{\beta}(\tau) + \frac{\lambda(\zeta)}{(f'(\zeta))^2}, \tag{3.93}$$

where $\lambda(\cdot)$ satisfies:

$$\lambda(\bar{\zeta} + 1) - \lambda(\bar{\zeta}) = 0. \tag{3.94}$$

Proof Using the change of variables (3.87), (3.88) and introducing a new function Z by means of

$$Y = \pi \cot(\pi(\zeta - \bar{\zeta} - Z)) \tag{3.95}$$

we transform (3.75–3.76) into

$$Z_\zeta = \frac{\sin^2(\pi(\zeta - \bar{\zeta} - Z))}{\pi^2}\lambda(\zeta), \tag{3.96}$$

$$Z(\zeta; \bar{\zeta}) = Z(\zeta; \bar{\zeta} + 1) = 0, \tag{3.97}$$

where we define $\lambda(\cdot)$ as:

$$\lambda(\zeta) := (\beta(\tau) - \bar{\beta}(\tau))(f'(\zeta))^2.$$

Integrating (3.96) in the interval $(\bar{\zeta}, \bar{\zeta} + 1)$ and using (3.97) we obtain

$$\int_{\bar{\zeta}}^{\bar{\zeta}+1} \sin^2(\pi(\zeta - \bar{\zeta} - Z(\zeta; \bar{\zeta})))\lambda(\zeta)d\zeta = 0. \tag{3.98}$$

Differentiating (3.98) we arrive at:

$$\int_{\bar{\zeta}}^{\bar{\zeta}+1} \sin(2\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))\left(1 + \frac{\partial Z(\zeta, \bar{\zeta})}{\partial \bar{\zeta}}\right)\lambda(\zeta)d\zeta = 0. \tag{3.99}$$

We can compute $\frac{\partial Z(\zeta,\bar{\zeta})}{\partial \bar{\zeta}}$ differentiating (3.96), (3.97) with respect to $\bar{\zeta}$. Then:

$$\left(\frac{\partial Z}{\partial \bar{\zeta}}\right)_\zeta = -\frac{\sin(2\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))\lambda(\zeta)}{\pi}\left(1 + \frac{\partial Z}{\partial \bar{\zeta}}\right), \tag{3.100}$$

$$\frac{\partial Z}{\partial \bar{\zeta}}(\bar{\zeta}, \bar{\zeta}) = \frac{\partial Z}{\partial \bar{\zeta}}(\bar{\zeta} + 1, \bar{\zeta}) = 0. \tag{3.101}$$

Solving (3.100), (3.101) we obtain:

$$\frac{\partial Z}{\partial \bar{\zeta}}(\zeta, \bar{\zeta}) = -\int_{\bar{\zeta}}^{\zeta} e^{-\bar{\psi}(\zeta,\bar{\zeta})+\bar{\psi}(\eta,\bar{\zeta})} \frac{\partial \psi(\eta, \bar{\zeta})}{\partial \zeta} d\eta = (e^{-\bar{\psi}(\zeta,\bar{\zeta})} - 1), \tag{3.102}$$

where

$$\psi(\zeta, \bar{\zeta}) := \frac{1}{\pi} \int_{\bar{\zeta}}^{\zeta} \sin(2\pi(\eta - \bar{\zeta} - Z(\eta, \bar{\zeta})))\lambda(\eta) d\eta. \tag{3.103}$$

Using (3.102) in (3.99) we deduce that:

$$0 = \int_{\bar{\zeta}}^{\bar{\zeta}+1} \frac{\partial \psi(\zeta, \bar{\zeta})}{\partial \zeta} e^{-\psi(\zeta,\bar{\zeta})} d\zeta = e^{-\psi(\bar{\zeta}+1,\bar{\zeta})} - 1$$

or, equivalently

$$\psi(\bar{\zeta} + 1, \bar{\zeta}) = \frac{1}{\pi} \int_{\bar{\zeta}}^{\bar{\zeta}+1} \sin(2\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))\lambda(\zeta) d\zeta = 0. \tag{3.104}$$

Differentiating (3.104) with respect to $\bar{\zeta}$ and using again (3.102) we arrive at

$$\int_{\bar{\zeta}}^{\bar{\zeta}+1} \cos(2\pi(\zeta - \bar{\zeta} - Z(\zeta, \bar{\zeta})))e^{-\psi(\zeta,\bar{\zeta})}\lambda(\zeta) d\zeta = 0. \tag{3.105}$$

Differentiating once more (3.105) with respect to $\bar{\zeta}$ and using (3.102), (3.104) we obtain

$$\lambda(\bar{\zeta} + 1) - \lambda(\bar{\zeta}) = \int_{\bar{\zeta}}^{\bar{\zeta}+1} K(\eta, \bar{\zeta})\lambda(\eta) d\eta, \tag{3.106}$$

where

$$\begin{aligned} K(\eta, \bar{\zeta}) := \Bigg[& 2\pi \sin(2\pi(\eta - \bar{\zeta} - Z(\eta, \bar{\zeta})))e^{-2\psi(\eta,\bar{\zeta})} \\ & + \cos(2\pi(\eta - \bar{\zeta} - Z(\eta, \bar{\zeta})))e^{-\psi(\eta,\bar{\zeta})} \frac{\partial \psi(\eta, \bar{\zeta})}{\partial \bar{\zeta}} \Bigg]. \end{aligned}$$

Using (3.103) it then follows that:

$$\begin{aligned} \int_{\bar{\zeta}}^{\bar{\zeta}+1} K(\eta, \bar{\zeta})\lambda(\eta) d\eta = \int_{\bar{\zeta}}^{\bar{\zeta}+1} \Bigg[& 2\pi \sin(2\pi(\eta - \bar{\zeta} - Z(\eta, \bar{\zeta})))e^{-2\psi(\eta,\bar{\zeta})} \\ & + \cos(2\pi(\eta - \bar{\zeta} - Z(\eta, \bar{\zeta})))e^{-\psi(\eta,\bar{\zeta})} \frac{\partial \psi(\eta, \bar{\zeta})}{\partial \bar{\zeta}} \Bigg] \lambda(\eta) d\eta \\ = 2\pi^2 \int_{\bar{\zeta}}^{\bar{\zeta}+1} & \frac{\partial \psi(\eta, \bar{\zeta})}{\partial \zeta} e^{-2\psi(\eta,\bar{\zeta})} d\eta \end{aligned}$$

$$-\frac{1}{\pi}\int_{\bar{\zeta}}^{\bar{\zeta}+1} d\eta\lambda(\eta)\cos(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))e^{-\psi(\eta,\bar{\zeta})}$$
$$\times\int_{\bar{\zeta}}^{\eta}\cos(2\pi(\xi-\bar{\zeta}-Z(\xi,\bar{\zeta})))(1+\frac{\partial Z}{\partial\bar{\zeta}}(\xi,\bar{\zeta}))\lambda(\xi)d\xi$$

whence, using (3.102), (3.104), (3.105) we can then write:

$$\int_{\bar{\zeta}}^{\bar{\zeta}+1} K(\eta,\bar{\zeta})\lambda(\eta)d\eta$$
$$=\pi^2\big(1-e^{-2\psi(\bar{\zeta}+1,\bar{\zeta})}\big)-\frac{1}{2\pi}\left(\int_{\bar{\zeta}}^{\bar{\zeta}+1}\cos(2\pi(\eta-\bar{\zeta}-Z(\eta,\bar{\zeta})))e^{-\psi(\eta,\bar{\zeta})}\lambda(\eta)d\eta\right)^2=0.$$

Plugging this formula into (3.106) we finally obtain (3.94) and the Theorem follows. □

Notice that the representation formula for the solutions of (3.75), (3.76) given in (3.94) shows that the approximate transition problem has a very large degree of nonuniqueness. Actually, this large degree of nonuniqueness is the same as the degree of nonuniqueness in the choice of $f(\cdot)$ (cf. (3.77)). Moreover, formulas (3.93), (3.94) shows that two solutions $\beta(\tau)$ of (3.75), (3.76) differ in a function $\frac{\lambda(\zeta)}{(f'(\zeta))^2}$. Actually this difference of terms is very small compared with the leading order asymptotics of the function $\beta(\tau)$ as $\tau\to\infty$. More precisely, for functions S satisfying (3.108) the term $\frac{\lambda(\zeta)}{(f'(\zeta))^2}$ is basically the correction "beyond all the orders" whose analysis plays a crucial role in [14]. The analysis in this Subsection avoids the use of explicit asymptotics for the function S and provides a much simpler description of the transition of the characteristic curves across the critical line that the one in that paper.

In the rest of this paper we approximate the dynamics of both the transition problem (3.53–3.57) and the whole LSW dynamics as perturbations of (3.94). The main result of this paper is the possibility of describing the LSW dynamics as a perturbation of the explicitly solvable problem studied in this Subsection.

3.4.3.3 Some Admissible Asymptotics for the Initial Data Let us mention a particular class of functions S that we will use repeatedly in the following. Suppose that G_0 satisfies:

$$G_0(W)\sim CW^B(\log(W))^De^{-W^A}\quad\text{as } W\to\infty,\ A>0,\ B,D\in\mathbb{R}. \tag{3.107}$$

Using (3.56) we then obtain:

$$\omega(\tau)\sim(\tau)^a\quad\text{as }\tau\to\infty,\ a=\frac{1}{A}>0.$$

It then follows from (3.74) that

$$S(\tau)\sim a\log(\tau)\quad\text{as }\tau\to\infty. \tag{3.108}$$

We notice, for further reference that the functions with the asymptotics (3.107) satisfy:

$$\frac{G_0(w_{\text{ext}}(-X_0-\lambda(X_0)\zeta))}{G_0(w_{\text{ext}}(-X_0))}=e^{-\zeta(1+\delta(X_0,\zeta))}\quad\text{as } X_0,\to\infty, \tag{3.109}$$

where w_{ext} is as in (3.59) $\lim_{X_0\to\infty} \delta(X_0,\zeta) \to 0$ uniformly in compact sets of ζ and:

$$\lambda(X_0) \sim \frac{1}{A}\frac{1}{(X_0)^A} \quad \text{as } X_0 \to \infty.$$

3.4.3.4 On the Asymptotics of the Function $\beta(\tau)$ It is interesting to verify that the function $\beta(\tau)$ defined by means of (3.85), (3.93) has the asymptotic behaviours computed in [6, 14] for functions S satisfying (3.108). Indeed, differentiating (3.77) three times and using (3.108) we obtain the approximations:

$$af'(\zeta+1) = f(\zeta+1)f'(\zeta), \tag{3.110}$$

$$\frac{f''(\zeta+1)}{f'(\zeta+1)} = \frac{f''(\zeta)}{f'(\zeta)} + \frac{1}{a}f'(\zeta), \tag{3.111}$$

$$\frac{f'''(\zeta+1)}{f'(\zeta+1)} = \frac{f'''(\zeta)}{f'(\zeta)} + \frac{3}{a}f''(\zeta) + \frac{1}{a^2}(f'(\zeta))^2. \tag{3.112}$$

Using (3.111), (3.112) in (3.84) we obtain the following transformation law for the Schwartzian derivative:

$$\{f,\zeta\} = -\frac{1}{2a^2}(f'(\zeta-1))^2 + \{f,\zeta-1\}$$

and plugging this formula into (3.85) we obtain, using also (3.110):

$$\bar{\beta}(f(\zeta)) = -\frac{1}{4}\frac{1}{(f(\zeta))^2} + \left(\frac{f'(\zeta-1)}{f'(\zeta)}\right)^2 \bar{\beta}(f(\zeta-1))$$

whence, using (3.77), (3.87), (3.110):

$$\bar{\beta}(\tau) = -\frac{1}{4}\frac{1}{(\tau)^2} + \left(\frac{a}{\tau}\right)^2 \bar{\beta}(S(\tau)). \tag{3.113}$$

Iterating this formula we obtain the asymptotics:

$$\bar{\beta}(\tau) \sim -\frac{1}{4}\left[\frac{1}{(\tau)^2} + \frac{1}{(\tau)^2(\log(\tau))^2} + \frac{1}{(\tau)^2(\log(\tau))^2(\log(\log(\tau)))^2} + \cdots\right] \tag{3.114}$$

as $\tau \to \infty$. This asymptotics has been obtained in [6, 14] using different methods. Let us remark that the difference between the functions $\beta(\tau)$ and $\bar{\beta}(\tau)$ differs is a term smaller than all the terms in the series on the right hand side of (3.114). Indeed, due to (3.94) it follows that λ is bounded. Therefore, (3.93) implies that the difference $\beta(\tau) - \bar{\beta}(\tau)$ is of order $\frac{1}{(f'(\zeta))^2}$. Due to (3.110) we have that this term must be understood heuristically as

$$\frac{1}{(f'(\zeta))^2} \sim \frac{1}{(\tau\log(\tau)\log\log(\tau)\dots)^2} \tag{3.115}$$

as $\tau \to \infty$, whence it is smaller than all the terms in (3.114). Corrective terms analogous to (3.115) have been computed in [14] using more involved, explicit computations.

Notice that the asymptotics (3.115) cannot be understood in its strict mathematical sense, but only in a heuristic way, because it is impossible to take a large number of logarithmic functions of a positive number τ without obtaining negative numbers. A longer, but somehow more precise way of giving a meaning to the asymptotics of a function like $h_{1,n}(\tau) = \log\log\log ..^{(n)} .. \log(\tau)$ is replacing it by $\bar{h}_{1,n}(\tau) = \log(1 + \log(1 + \log(1 + ..^{(n)} ..(1 + \log(\tau + 1)))))$. In both cases we take n iterations of a basic functional block that is $\log(\cdot)$ in the first case and $\log(1 + \cdot)$ in the second. For any fixed value of n we have $h_{1,n}(\tau) \sim \bar{h}_{1,n}(\tau)$ as $\tau \to \infty$. The main advantage of the function $\bar{h}_{1,n}(\tau)$ is that it can be defined for any $\tau \geq 0$, while $h_{1,n}(\tau)$ is defined only for large values of τ. Actually this was the approach used in [14] to give a meaning to the iterated logarithmic functions arising in formulas like (3.115).

The main advantage of using the function $f(\zeta)$ defined by means of (3.77), (3.78) in order to study the transition problem is that it avoids the detailed study of the iterated logarithmic functions made in [14]. On the other hand, the argument above explains in an alternative manner the onset of these iterated logarithmic series in [6, 14].

3.4.4 *Long Time Asymptotics for the Transition Problem*

In this Subsection we study the long time asymptotics for the transition problem (3.53–3.57). We will assume from now on that $\omega(\tau)$ in (3.56) is as differentiable as required by the computations.

Some of the estimates formally derived in this and in the next Subsection have been rigorously proved in [15], where a different, simpler "transition problem" for the characteristic curves near the critical line $W = 1$ has been studied using related ideas.

Our goal is to approximate this problem by means of the approximate transition problem (3.75), (3.76). Let us denote as $\bar{\beta}(\tau)$ a solution of this problem given by (3.87), (3.85).

To study the long time asymptotics of the solutions of (3.53–3.57) we use a change of variables similar to (3.87), (3.88), (3.95), namely:

$$\tau = f(\zeta), \tag{3.116}$$

$$W - 1 = \frac{3Y}{f'(\zeta)} + \frac{3f''(\zeta)}{2(f'(\zeta))^2}, \tag{3.117}$$

$$Y = \pi \cot(\pi(\zeta - \bar{\zeta} - Z)), \tag{3.118}$$

where $Z = Z(\zeta, \bar{\zeta}, \zeta_0)$, $\zeta_0 = f^{-1}(T_0)$ and $\bar{\zeta} = f^{-1}(\bar{\tau}) - 1$ and T_0 is the starting point where we begin solving the transition problem (cf. (3.78)) These changes transform (3.53–3.57) into:

$$Z_\zeta = \frac{\sin^2(\pi(\zeta - \bar{\zeta} - Z))}{\pi^2}[\lambda(\zeta) + R(Z, \zeta, \bar{\zeta}, \zeta_0)], \tag{3.119}$$

$$Z = -\frac{1}{\pi} \operatorname{arccot}\left(\frac{f'(\zeta_0)}{3\pi}\left(\omega(\bar{\tau}) - 1 - \frac{3f''(\zeta_0)}{2(f'(\zeta_0))^2}\right)\right), \quad \zeta = \zeta_0, \tag{3.120}$$

$$Z = \frac{1}{\pi} \operatorname{arccot}\left(\frac{f'(\bar{\zeta}+1)}{3\pi}\left[\frac{1}{2} + \frac{3f''(\bar{\zeta}+1)}{2(f'(\bar{\zeta}+1))^2}\right]\right), \quad \zeta = \bar{\zeta} + 1, \tag{3.121}$$

where we have used that $\alpha = \frac{1}{2}$ and we define:

$$\mu(\zeta) = \beta(\tau) - \bar{\beta}(\tau), \tag{3.122}$$

$$\lambda(\zeta) = \mu(\zeta)(f'(\zeta))^2, \tag{3.123}$$

$$R(Z,\zeta,\bar{\zeta},\zeta_0)=(f'(\zeta))^2\left[\frac{h_1(W)}{3}+\beta(f(\zeta))h_2(W)\right],\tag{3.124}$$

$$h_1(W)=\frac{(W-1)^2}{3}-2+3W^{1/3}-W,\tag{3.125}$$

$$h_2(W)=W^{1/3}-1.\tag{3.126}$$

The problem (3.119–3.121) might be considered as a perturbation of the exactly solvable problem (3.96), (3.97). It turns out, however, that the presence of the boundary conditions (3.120), (3.121) instead of (3.97) introduce some "boundary effects" that we pass to discuss in detail.

Many of the computations below are valid for rather general functions $S(\cdot)$. Nevertheless, we will assume by definiteness that S satisfies (3.108). This includes in particular initial data like the ones in (3.107). Under this assumption we can think on the function $f(\zeta)$ in (3.87) as

$$f(\zeta)\approx\exp\left(\frac{1}{a}\exp\left(\cdots\left(\frac{1}{a}\exp\left(\frac{\zeta}{a}\right)\right)\right)\right)\quad\text{as }\zeta\to\infty.$$

Several properties of the function f and its derivatives under the assumption (3.108) are collected in Appendix 2.

Integrating (3.119) and using the boundary conditions (3.120), (3.121) we obtain:

$$\begin{aligned}&\int_{\zeta_0}^{\bar{\zeta}+1}\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z(\zeta,\bar{\zeta},\zeta_0)))}{\pi^2}[\lambda(\zeta)+R(Z(\zeta,\bar{\zeta},\zeta_0),\zeta,\bar{\zeta},\zeta_0)]d\zeta\\&\quad=\theta(\bar{\zeta},\zeta_0)-(\zeta_0-\bar{\zeta}),\end{aligned}\tag{3.127}$$

where

$$\begin{aligned}\theta(\bar{\zeta},\zeta_0):=&\ \frac{1}{\pi}\operatorname{arccot}\left(\frac{f'(\bar{\zeta}+1)}{3\pi}\left[\frac{1}{2}+\frac{3f''(\bar{\zeta}+1)}{2(f'(\bar{\zeta}+1))^2}\right]\right)\\&+\frac{1}{\pi}\operatorname{arccot}\left(\frac{f'(\zeta_0)}{3\pi}\left(\omega(\bar{\tau})-1-\frac{3f''(\zeta_0)}{2(f'(\zeta_0))^2}\right)\right).\end{aligned}$$

Equation (3.127) is reminiscent of (3.98). Actually, (3.127) can be transformed to a form closer to (3.98) by means of some algebraic manipulations. Using the trigonometric identity $\sin^2(x)=\frac{1}{1+\cot^2(x)}$ as well as (3.118) we obtain:

$$\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z(\zeta,\bar{\zeta},\zeta_0)))}{\pi^2}=\frac{9}{9\pi^2+(f'(\zeta))^2\left(W-1-\frac{3f''(\zeta)}{2(f'(\zeta))^2}\right)^2}.\tag{3.128}$$

Therefore, the second term on the left of (3.127) becomes:

$$\begin{aligned}&\int_{\zeta_0}^{\bar{\zeta}+1}\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z(\zeta,\bar{\zeta},\zeta_0)))}{\pi^2}R(Z(\zeta,\bar{\zeta},\zeta_0),\zeta,\bar{\zeta},\zeta_0)d\zeta-(\bar{\zeta}-\zeta_0)\\&\quad=\int_{\zeta_0}^{\bar{\zeta}+1}\left[\frac{3(f'(\zeta))^2h_1(W)}{9\pi^2+(f'(\zeta))^2\left(W-1-\frac{3f''(\zeta)}{2(f'(\zeta))^2}\right)^2}-\chi(\bar{\zeta}-\zeta)\right]d\zeta\end{aligned}$$

$$+\int_{\zeta_0}^{\bar{\zeta}+1} \frac{9(f'(\zeta))^2\beta(f(\zeta))h_2(W)}{9\pi^2+(f'(\zeta))^2\big(W-1-\frac{3f''(\zeta)}{2(f'(\zeta))^2}\big)^2}d\zeta$$

$$:= I_1 + I_2,$$

where $\chi(s)=1,\ s\geq 0,\ \chi(s)=0,\ s<0$, and $h_1(W)$, $h_2(W)$ are as in (3.125), (3.126).

Since $\beta(\tau)\to 0$ we can approximate the function W in the integrals $I_1,\ I_2$ as

$$W \sim w_{\text{ext}}(f(\zeta)-f(\bar{\zeta})) \tag{3.129}$$

for $\zeta_0 \leq \zeta \leq \bar{\zeta}+\delta_1(\bar{\zeta})$, where $\delta_1(\bar{\zeta}) \sim \frac{1}{(f'(\bar{\zeta}))^\gamma}$ where from now on γ is a generic positive constant. Moreover:

$$W \sim w_{\text{int}}(f(\zeta)-f(\bar{\zeta}+1)-F_{\text{int}}(\alpha)) \tag{3.130}$$

for $(\bar{\zeta}+1)-\delta_2(\bar{\zeta}) \leq \zeta \leq \bar{\zeta}+1$, where $\delta_2(\bar{\zeta}) \sim \frac{1}{(f'(\bar{\zeta}+1))^\gamma}$. Due to the fast growth of the function $f(\zeta)$ as $\zeta\to\infty$, it turns out that the main contribution to the integrals $I_1,\ I_2$ is due to the intervals $[\zeta_0, \bar{\zeta}+\delta_1(\bar{\zeta})]$. Then:

$$I_1 \sim \int_{\zeta_0}^{\bar{\zeta}+\delta_1(\bar{\zeta})} \Phi_1(f(\zeta)-f(\bar{\zeta}))d\zeta,$$

$$I_2 \sim \int_{\zeta_0}^{\bar{\zeta}+\delta_1(\bar{\zeta})} \Phi_2(f(\zeta)-f(\bar{\zeta},))d\zeta,$$

where

$$\Phi_1(f(\zeta)-f(\bar{\zeta})) := \left[\frac{3(f'(\zeta))^2h_1(w_{\text{ext}}(f(\zeta)-f(\bar{\zeta})))}{9\pi^2+(f'(\zeta))^2\big(w_{\text{ext}}(f(\zeta)-f(\bar{\zeta}))-1-\frac{3f''(\zeta)}{2(f'(\zeta))^2}\big)^2}-\chi(\bar{\zeta}-\zeta)\right],$$

$$\Phi_2(f(\zeta)-f(\bar{\zeta})) := \frac{9(f'(\zeta))^2\beta(f(\zeta))h_2(w_{\text{ext}}(f(\zeta)-f(\bar{\zeta})))}{9\pi^2+(f'(\zeta))^2\big(w_{\text{ext}}(f(\zeta)-f(\bar{\zeta}))-1-\frac{3f''(\zeta)}{2(f'(\zeta))^2}\big)^2}.$$

Notice that, due to the exponential decay of $w_{\text{ext}}(s)$, $\Phi_1(f(\zeta)-f(\bar{\zeta}))$ decays exponentially as $(\bar{\zeta}-\zeta)\to\infty$, and $|\Phi_1(f(\zeta)-f(\bar{\zeta}))| \leq \frac{C}{f'(\bar{\zeta})}\frac{1}{(\zeta-\bar{\zeta})^3}$ as $(\zeta-\bar{\zeta})\to\infty$. Using the properties of $f(\zeta)$ in Appendix 2 it follows that:

$$\int_{\zeta_0}^{\bar{\zeta}+\delta_1(\bar{\zeta})} \Phi_1(f(\zeta)-f(\bar{\zeta}))d\zeta \sim \frac{1}{f'(\bar{\zeta})}\int_{-\infty}^{\infty}\Phi_1(s-f(\bar{\zeta}))ds := \frac{a}{f'(\bar{\zeta})}.$$

On the other hand, since $\beta(f(\zeta))$ decays like $\frac{1}{(f'(\bar{\zeta}))^{1+\gamma}}$ we obtain that the term I_2 might be estimated in a similar manner. Similar estimates can be obtained for the derivatives of I_1, I_2 (cf. [15]). Therefore, (3.127) can be approximated as:

$$\int_{\zeta_0}^{\bar{\zeta}+1} \frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z(\zeta,\bar{\zeta},\zeta_0)))\lambda(\zeta)}{\pi^2}d\zeta = \theta(\bar{\zeta},\zeta_0)-\varepsilon(\bar{\zeta}), \tag{3.131}$$

where $|\varepsilon(\bar{\zeta})| \leq \frac{C}{f'(\bar{\zeta})}$.

We have then reduced the study of the transition problem to the integral equation (3.131). Notice that (3.128) implies that the kernel $\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z(\zeta,\bar{\zeta},\zeta_0)))}{\pi^2}$ becomes negligible as soon as $(W-1)$ becomes of order one. This suggests the following rough approximation for (3.131):

$$\int_{\bar{\zeta}}^{\bar{\zeta}+1} \frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z(\zeta,\bar{\zeta},\zeta_0)))\lambda(\zeta)}{\pi^2} d\zeta = \theta(\bar{\zeta},\zeta_0) - \varepsilon(\bar{\zeta},\zeta_0). \tag{3.132}$$

This equation is a nonhomogeneous version of (3.98). Notice that the right-hand side of (3.132) is bounded by $\frac{C}{(f'(\bar{\zeta}))^\gamma}$, and similar estimates can be obtained for the derivatives (cf. [15]). Arguing then as in Sect. 3.4.3, and including in the error terms the contributions of R in (3.119), we can transform (3.132) into

$$\lambda(\bar{\zeta}+1) - \lambda(\bar{\zeta}) = O\left(\frac{1}{(f'(\bar{\zeta}))^\gamma}\right) \tag{3.133}$$

as $\bar{\zeta}\to\infty$. This equation might be solved by iteration. Due to the fast growth of the function $f'(\bar{\zeta})$ the resulting function $\lambda(\bar{\zeta})$ is globally bounded. More precisely, the function $\lambda(\bar{\zeta})$ approaches asymptotically to a periodic function with period one as $\bar{\zeta}\to\infty$. Due to (3.122), (3.123) it follows that $\tilde{\beta}(\tau)$ and $\bar{\beta}(\tau)$ differ in a very small function as $\tau\to\infty$.

The approximation of analysis of (3.131) by means of (3.132) must be made more carefully because the three differentiations of (3.132) require to differentiate (3.128) and this yields terms containing derivatives of λ that must be examined in detail.

More precisely, differentiating (3.131) and using (3.128) we obtain, to the leading order:

$$\begin{aligned}
&\frac{d(\theta(\bar{\zeta},\zeta_0)-\varepsilon(\bar{\zeta},\zeta_0))}{d\bar{\zeta}} \\
&\quad = \frac{9}{(f'(\bar{\zeta}+1))^2}\frac{\lambda(\bar{\zeta}+1)}{(\frac{1}{2})^2} + \int_{\zeta_0}^{\bar{\zeta}+1} \frac{d}{d\bar{\zeta}}\left(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\right)\lambda(\zeta)d\zeta.
\end{aligned} \tag{3.134}$$

Combining (3.128), (3.130) we derive the following approximation for $\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}$ in the region $\zeta\approx\bar{\zeta}+1$:

$$\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2} \sim \frac{9\Omega(f(\zeta)-f(\zeta)-F_{\text{int}}(\alpha))}{(f'(\zeta))^2},$$

where:

$$\Omega(s) = \frac{1}{(w_{\text{int}}(s)-1)^2}.$$

Therefore, we derive the following approximations, for the derivatives of $\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}$ in the region $\zeta\approx\bar{\zeta}+1$:

$$\begin{aligned}
&\frac{d}{d\bar{\zeta}}\left(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\right) \sim -\frac{9\Omega'(f(\zeta)-f(\bar{\zeta}+1)-F_{\text{int}}(\alpha))}{f'(\zeta)}, \\
&\frac{d^2}{d\bar{\zeta}^2}\left(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\right) \sim 9\Omega''(f(\zeta)-f(\bar{\zeta}+1)-F_{\text{int}}(\alpha)), \\
&\frac{d^3}{d\bar{\zeta}^3}\left(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\right) \sim -9f'(\bar{\zeta}+1)\Omega''(f(\zeta)-f(\bar{\zeta}+1)-F_{\text{int}}(\alpha)),
\end{aligned}$$

where we have used the properties of the function f in Appendix 2. Differentiating (3.134), using the previous approximations for the derivatives of $\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}$, and neglecting the smaller contributions multiplying the terms $\lambda(\bar{\zeta}+1)$ or its derivatives we arrive at:

$$\begin{aligned}
&\frac{d^2(\theta(\bar{\zeta},\zeta_0)-\varepsilon(\bar{\zeta},\zeta_0))}{d\bar{\zeta}^2}\\
&\quad=\frac{9}{(f'(\bar{\zeta}+1))^2}\frac{\lambda'(\bar{\zeta}+1)}{(\frac{1}{2})^2}-\frac{9\Omega'(-F_{\text{int}}(\frac{1}{2}))}{f'(\bar{\zeta}+1)}\lambda(\bar{\zeta}+1)\\
&\qquad+\int_{\zeta_0}^{\bar{\zeta}+1}\frac{d^2}{d\bar{\zeta}^2}\left(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\right)\lambda(\zeta)d\zeta,
\end{aligned}\tag{3.135}$$

$$\begin{aligned}
&\frac{d^3(\theta(\bar{\zeta},\zeta_0)-\varepsilon(\bar{\zeta},\zeta_0))}{d\bar{\zeta}^3}\\
&\quad=\frac{9}{(f'(\bar{\zeta}+1))^2}\frac{\lambda''(\bar{\zeta}+1)}{(\frac{1}{2})^2}-\frac{9\Omega'(-F_{\text{int}}(\frac{1}{2}))}{f'(\bar{\zeta}+1)}\lambda'(\bar{\zeta}+1)+9\Omega''\left(-F_{\text{int}}\left(\frac{1}{2}\right)\right)\lambda(\bar{\zeta}+1)\\
&\qquad+\int_{\zeta_0}^{\bar{\zeta}+1}\frac{d^3}{d\bar{\zeta}^3}\left(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\right)\lambda(\zeta)d\zeta.
\end{aligned}\tag{3.136}$$

We now derive approximations for the terms $\frac{d^2}{d\bar{\zeta}^2}(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2})$, $\frac{d^3}{d\bar{\zeta}^3}(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2})$ in (3.136), (3.135). In order to compute an approximation for these terms in the region $\zeta\approx\bar{\zeta}$ we use (3.128), (3.129). Then:

$$\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\sim\frac{9\Psi(f(\zeta)-f(\bar{\zeta}))}{(f'(\zeta))^2}$$

and differentiating this formula we then arrive at:

$$\begin{aligned}
\frac{d}{d\bar{\zeta}}\left(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\right)&\sim-\frac{9\Psi'(f(\zeta)-f(\bar{\zeta}))}{f'(\zeta)},\\
\frac{d^2}{d\bar{\zeta}^2}\left(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\right)&\sim 9\Psi''(f(\zeta)-f(\bar{\zeta})),\\
\frac{d^3}{d\bar{\zeta}^3}\left(\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\right)&\sim -9f'(\zeta)\Psi''(f(\zeta)-f(\bar{\zeta})).
\end{aligned}$$

On the other hand, in order to estimate the term $\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}$ in the region $[\bar{\zeta}+\delta_1(\bar{\zeta}),\bar{\zeta}+1-\delta_2(\bar{\zeta})]$ we will assume that the term Z might be neglected and therefore

$$\frac{\sin^2(\pi(\zeta-\bar{\zeta}-Z))}{\pi^2}\sim\frac{\sin^2(\pi(\zeta-\bar{\zeta}))}{\pi^2}.\tag{3.137}$$

This approximation is useful to simplify the computations. Its validity will be discussed below in detail. Using (3.137), (3.135) becomes:

$$
\begin{aligned}
&\frac{d^3(\theta(\bar{\zeta},\zeta_0)-\varepsilon(\bar{\zeta},\zeta_0))}{d\bar{\zeta}^3}\\
&\quad=\frac{9}{(f'(\bar{\zeta}+1))^2}\frac{\lambda''(\bar{\zeta}+1)}{(\frac{1}{2})^2}-\frac{9\Omega'(-F_{\mathrm{int}}(\frac{1}{2}))}{f'(\bar{\zeta}+1)}\lambda'(\bar{\zeta}+1)+9\Omega''\left(-F_{\mathrm{int}}\left(\frac{1}{2}\right)\right)\lambda(\bar{\zeta}+1)\\
&\qquad-9\int_{\bar{\zeta}+1-\delta_2(\bar{\zeta})}^{\bar{\zeta}+1}f'(\zeta)\Omega'''\left(f(\zeta)-f(\bar{\zeta}+1)-F_{\mathrm{int}}\left(\frac{1}{2}\right)\right)\lambda(\zeta)d\zeta\\
&\qquad-9\int_{\zeta_0}^{\bar{\zeta}+\delta_1(\bar{\zeta})}f'(\zeta)\Psi'''(f(\zeta)-f(\bar{\zeta}))\lambda(\zeta)d\zeta\\
&\qquad-4\pi^2\int_{\bar{\zeta}}^{\bar{\zeta}+1}\sin(2\pi(\zeta-\bar{\zeta}))\lambda(\zeta)d\zeta. \qquad (3.138)
\end{aligned}
$$

In order to estimate the term $\int_{\bar{\zeta}}^{\bar{\zeta}+1}\sin(2\pi(\zeta-\bar{\zeta}))\lambda(\zeta)d\zeta$ in (3.138) we use similar approximations in (3.134). Plugging the resulting formula into (3.138) to obtain:

$$
\begin{aligned}
&\frac{1}{(f'(\bar{\zeta}+1))^2}\frac{\lambda''(\bar{\zeta}+1)}{(\frac{1}{2})^2}-\frac{\Omega'(-F_{\mathrm{int}}(\frac{1}{2}))}{f'(\bar{\zeta}+1)}\lambda'(\bar{\zeta}+1)+\Omega''\left(-F_{\mathrm{int}}\left(\frac{1}{2}\right)\right)\lambda(\bar{\zeta}+1)\\
&\quad-\int_{\zeta_0}^{\bar{\zeta}+1}f'(\zeta)\Omega'''\left(f(\zeta)-f(\bar{\zeta}+1)-F_{\mathrm{int}}\left(\frac{1}{2}\right)\right)\lambda(\zeta)d\zeta\\
&\quad-\int_{\zeta_0}^{\bar{\zeta}+\delta_1(\bar{\zeta})}f'(\zeta)\Psi'''(f(\zeta)-f(\bar{\zeta}))\lambda(\zeta)d\zeta\\
&\quad=O\left(\frac{1}{(f(\bar{\zeta}))^{\gamma}}\right) \qquad (3.139)
\end{aligned}
$$

as $\bar{\zeta}\to\infty$. We have used the fast decay of the term $\Omega'''(s)$, in order to replace the lower extreme of integration in the first integral by ζ_0.

The integrodifferential equation (3.139) is the basic equation that describes the evolution of the function $\lambda(\zeta)$, or equivalently $\beta(\tau)$. More precisely, given $\omega(\tau)$ as in (3.56) we determine $\beta(\tau)$ as follows. First we compute $S(\tau)$ using (3.74). We can then compute the functions $f(\zeta)$ and $\bar{\beta}(\tau)$ using (3.79) and (3.85). Solving (3.139) we obtain the function $\lambda(\zeta)$ that allows to compute $\beta(\tau)$ by means of (3.122), (3.123).

Equation (3.139) has two different time scales as $\tau\to\infty$. Indeed, since $\frac{d}{d\tau}=\frac{1}{f'(\zeta)}\frac{d}{d\zeta}$, it follows that the natural time scale for the terms containing the derivatives is $\delta\zeta\approx\frac{1}{f(\bar{\zeta}+1)}$, or equivalently $\delta\tau\approx 1$. On the other hand, the integral terms might be approximated, for functions λ that are approximately constant in the time scale τ as Dirac masses in $\zeta=\bar{\zeta}$, $\zeta=\bar{\zeta}+1$ respectively. In particular these terms alone would transform (3.139) in a delay equation of the form (3.133). In the original time variable τ this delay equation would relate the values of λ at times τ and $S(\tau)$. In particular, under the assumption (3.108), $S(\tau)$ varies in a very slow, "adiabatic" manner in the time scale τ. It is shown in Appendix 3 that, if the second integral term is frozen, the remaining terms in (3.139) yield stabilization to steady states in times $\delta\zeta\approx\frac{1}{f(\bar{\zeta}+1)}$, or equivalently in times $\delta\tau\approx 1$. This implies that the function λ becomes approximately constant in time intervals or size $\delta\tau\approx 1$. Therefore, the integral terms might be approximated as Dirac masses and (3.139) becomes (3.133). The amount of mass concentrated in the Dirac masses can be computed using:

$$\Omega''(-F_{\text{int}}(\alpha)) - 9\int_{-\infty}^{0} \Omega'''(s - F_{\text{int}}(\alpha))ds = \int_{-\infty}^{\infty} \Psi'''(s)ds = \frac{2}{9}.$$

Let us discuss now the validity of the assumption (3.137). Notice that due to (3.119–3.121) the order of magnitude of Z is the same as the one of λ in the region where $\zeta - \bar{\zeta}$ is of order one. Nevertheless, due to the arbitrariness of $f(\zeta)$ would be possible to choose λ small for large ζ and (3.133) would imply that λ is small for large ζ. In any case, since the order of magnitude of λ is one, the argument above is not completely rigorous. However, the argument could be made keeping the term Z in (3.137) in the derivation of (3.139). In particular this would introduce several additional terms containing Z and its derivatives similar to the ones in Sect. 3.4.3. Detailed, although more technical computations, following this approach can be found in [15]. Nevertheless, the final result is again (3.139). Therefore the analysis above can be kept without major changes.

It is interesting to discuss the meaning of the two different time scales in the integrodifferential equation (3.139). The time scale $(\bar{\tau} - \tau)$ of order one in which stabilization takes place is associated to the effect of the characteristics that are placed in the region $W \in [0, 1)$. These characteristics yield stabilization of the dynamics of the LSW model to the simplified dynamics (3.133). This simplified equation indicates that in some sense, the solutions of the transition problem (3.53–3.57) are able to "see" the values of the function $\bar{\beta}(\bar{\tau})$ at the times $\tau = S(\bar{\tau})$, that is the time when the characteristic that reaches the value $W = \alpha$ for $\tau = \bar{\tau}$ arrived to that critical line.

From now on, in order to distinguish the characteristic curves associated to the whole LSW model and the trajectories associated to the solutions of (3.53), (3.54) with the function $\beta(\tau) = \tilde{\beta}(\tau)$ solving the transition problem we will denote from now on as $\bar{W}(\tau; W_0)$ the solutions of (3.53), (3.54) in this last case. We will denote as $\bar{W}_0(\tau; W)$ the inverse function of $\bar{W}(\tau; \cdot)$ for each fixed $\tau \geq 0$. Then:

$$\bar{W}(\tau; \bar{W}_0(\tau; W)) = W. \tag{3.140}$$

Notice that with this notation (3.55) becomes:

$$\bar{W}(\tau; \omega(\tau)) = \frac{1}{2}. \tag{3.141}$$

3.5 Linearizing the LSW Model Near Self-Similar Solutions

In this Section we will use ideas analogous to those in the previous subsection to linearize near self-similar solutions the whole LSW model and not just the transition problem (3.53–3.57).

We notice that the volume conservation condition (2.5), combined with the evolution of G along characteristic curves (3.14) is equivalent to

$$\int_0^{\infty} G(W, \tau)dW = e^{\tau} \int_0^{\infty} G_0(W_0(\tau; W))dW, \tag{3.142}$$

where the function $W_0(\tau; W)$ is as in (3.20).

On the other hand, using (3.24), we obtain

$$\begin{aligned} e^{\tau} \int_0^1 G_0(\omega(\tau + F_{\text{int}}(W) - F_{\text{int}}(\alpha) + \epsilon(\tau, W, \alpha)))dW + \int_1^{\infty} G(W, \tau)dW \\ = e^{\tau} \int_{W_0(\tau;0)}^{\infty} G_0(W_0(\tau; W))dW, \end{aligned} \tag{3.143}$$

where $\epsilon(\tau, W, \alpha)$ satisfies (3.22) and $\alpha = \frac{1}{2}$.

Combining (3.142), (3.143) we obtain:

$$\int_0^1 \left[G_0(W_0(\tau, W)) - G_0\left(\omega\left(\tau + F_{\text{int}}(W) - F_{\text{int}}\left(\frac{1}{2}\right) + \epsilon\left(\tau, W, \frac{1}{2}\right)\right)\right)\right] dW$$
$$= e^{-\tau} \int_1^\infty G(W, \tau) dW.$$

Using the first and the last terms in (3.24) it then follows that

$$\int_0^1 e^{-F_{\text{int}}(W)} \left[\frac{G_0(W_0(\tau, W))}{G_0(\omega(\tau + F_{\text{int}}(W) - F_{\text{int}}(\frac{1}{2}) + \epsilon(\tau, W, \frac{1}{2})))} - 1 \right] dW = \varepsilon(\tau), \tag{3.144}$$

where

$$\varepsilon(\tau) := (1 + o(1)) \int_1^\infty G(W, \tau) dW \quad \text{as } \tau \to \infty.$$

Notice that, the term $\int_1^\infty G(W, \tau) dW$ contains the mass of the particles in the supercritical region. For functions behaving in a self-similar manner as $\tau \to \infty$ this integral can be expected to be very small (actually exponentially small). We will then assume this in the rest of the argument and examine the contributions of the left-hand side of (3.144).

Suppose that $\bar{W}_0(\tau, W)$ is as in (3.140). It then follows from (3.21) and (3.141) that $\bar{W}_0(\tau, W) = \omega(\tau + F_{\text{int}}(W) - F_{\text{int}}(\frac{1}{2}) + \epsilon(\tau, W, \frac{1}{2}))$. On the other hand we will assume in all this Subsection that G_0 satisfies (3.109). Then (3.144) becomes, to the leading order:

$$\int_0^1 e^{-F_{\text{int}}(W)} \cdot \left[\exp\left(-\frac{W_0(\tau, W) - \bar{W}_0(\tau, W)}{\lambda(w_{\text{ext}}^{-1}(\bar{W}_0(\tau, W)))\bar{W}_0(\tau, W)} \right) - 1 \right] dW$$
$$= \varepsilon(\tau), \tag{3.145}$$

where we are using (3.58), (3.59) and where, from now on, we will include in $\varepsilon(\tau)$ additional corrective terms that converge to zero as $\tau \to \infty$. Notice that in the derivation of (3.145) we assume that the term inside the exponential is bounded. Moreover, we will assume also that $\frac{W_0(\tau,W)}{\bar{W}_0(\tau,W)} \to 1$. These assumptions will be justified "a posteriori" in a self-consistent manner, checking that the equations derived with these assumptions imply them. Linearizing then the exponential term we arrive at:

$$\int_0^1 \frac{e^{-F_{\text{int}}(W)}}{\lambda(w_{\text{ext}}^{-1}(\bar{W}_0(\tau, W)))} \left(\frac{W_0(\tau, W)}{\bar{W}_0(\tau, W)} - 1 \right) dW = \varepsilon(\tau). \tag{3.146}$$

In order to study the behaviour of the solutions of (4) we use again the change of variables (3.116–3.118). Let us denote from now on as $\bar{\lambda}(\zeta)$, $\bar{Z}(\zeta, \bar{\zeta}, \zeta_0)$ the approximated solution constructed in Sect. 3.4.4. Let us write $U := Z - \bar{Z}$. Linearizing in (3.119–3.121) it follows that:

$$U_\zeta = a(\zeta) U + \frac{\sin^2(\pi(\zeta - \bar{\zeta} - \bar{Z}))}{\pi^2} \sigma(\zeta),$$
$$U(\bar{\zeta} + 1) = 0,$$

where

$$a(\zeta) := \frac{\partial}{\partial Z} \left(\frac{\sin^2(\pi(\zeta - \bar{\zeta} - \bar{Z}))}{\pi^2} \right)\Bigg|_{Z=\bar{Z}} \bar{\lambda}(\zeta) + \frac{\partial}{\partial Z} \left(\frac{\sin^2(\pi(\zeta - \bar{\zeta} - \bar{Z})) R(Z, \zeta, \bar{\zeta})}{\pi^2} \right)\Bigg|_{Z=\bar{Z}}.$$

Our goal is to estimate $U(\zeta_0)$, that is given by:

$$U(\zeta_0) = -\int_{\zeta_0}^{\bar{\zeta}+1} e^{\int_\eta^{\bar{\zeta}+1} a(\xi)d\xi} \frac{\sin^2(\pi(\zeta - \bar{\zeta} - \bar{Z}))}{\pi^2}\sigma(\zeta)d\zeta. \tag{3.147}$$

It turns out that the integral term in (3.147) is small, because the term containing $R(Z,\zeta,\bar{\zeta})$ might be estimated as in Sect. 3.4.4, and the term containing $\bar{\lambda}(\zeta)$ can be estimated using the fact that $\bar{\lambda}$ is small. Therefore, to the leading order:

$$U(\zeta_0) = -\int_{\zeta_0}^{\bar{\zeta}+1} \frac{\sin^2(\pi(\zeta - \bar{\zeta} - \bar{Z}))}{\pi^2}\sigma(\zeta)d\zeta. \tag{3.148}$$

We now recall that our goal is to estimate the term in (4). To this end we use the change of variables (3.117), (3.118) that yields:

$$W_0(\tau, W) = 1 + \frac{3\pi\cot(\pi(\zeta - \bar{\zeta} - \bar{Z} - U))}{f'(\zeta)} + \frac{3f''(\zeta)}{2(f'(\zeta))^2} \quad \text{at } \zeta = \zeta_0,$$

$$\bar{W}_0(\tau, W) = 1 + \frac{3\pi\cot(\pi(\zeta - \bar{\zeta} - \bar{Z}))}{f'(\zeta)} + \frac{3f''(\zeta)}{2(f'(\zeta))^2} \quad \text{at } \zeta = \zeta_0.$$

In order to compute the difference $(W_0(\tau,W) - \bar{W}_0(\tau,W))$ we use the trigonometric formula $\cot(A-B) - \cot(A) = \frac{1+\cot^2(A)}{\cot(B)-\cot(A)}$. Then, after some computations we obtain:

$$\begin{aligned}
&\int_0^1 \frac{e^{-F_{\text{int}}(W)}}{\lambda(w_{\text{ext}}^{-1}(\bar{W}_0(\tau,W)))}\left(\frac{W_0(\tau,W)}{\bar{W}_0(\tau,W)} - 1\right)dW \\
&\quad = \int_0^1 \frac{e^{-F_{\text{int}}(W)}}{\lambda(w_{\text{ext}}^{-1}(\bar{W}_0(\tau,W)))}\frac{\cot(\pi(\zeta_0 - \bar{\zeta} - \bar{Z}))}{\cot(\pi(\zeta - \bar{\zeta} - \bar{Z})) - \cot(\pi U)}dW \\
&\quad = \varepsilon(\tau).
\end{aligned}$$

Notice that $\pi(\zeta_0 - \bar{\zeta} - \bar{Z}) \ll 1$. On the other hand we will assume, something to be checked "a posteriori" that $|U| \ll (\zeta_0 - \bar{\zeta} - \bar{Z})$. Then $\cot(\pi U) \gg \cot(\pi(\zeta - \bar{\zeta} - \bar{Z}))$, whence, to the leading order

$$\int_0^1 \frac{e^{-F_{\text{int}}(W)}}{\lambda(w_{\text{ext}}^{-1}(\bar{W}_0(\tau,W)))}\frac{\cot(\pi(\zeta_0 - \bar{\zeta} - \bar{Z}))}{\cot(\pi U)}dW = \varepsilon(\tau).$$

Using the formula $\cot(x) \sim \frac{1}{x}$ as $x \to 0$, it follows that:

$$\begin{aligned}
&\int_0^1 \left[e^{-F_{\text{int}}(W)}U(\zeta_0,\bar{\zeta},W)\right]dW \\
&\qquad = \frac{[-\varepsilon(\tau) + \tilde{\varepsilon}(\tau)]\lambda(w_{\text{ext}}^{-1}(\bar{W}_0(\tau,\alpha)))}{\pi\cot(\pi(\zeta_0 - \bar{\zeta} - \bar{Z}(\zeta_0,\bar{\zeta},\alpha)))} := \upsilon(\tau),
\end{aligned}$$

where U is given by (3.148) and where, from now on, $\upsilon(\tau)$ is a generic function satisfying $\upsilon(\tau) = O(\frac{1}{(f(\zeta))^\beta})$ as $\zeta \to \infty$, for some $\beta > 0$. Then:

$$\int_{\zeta_0}^{\bar{\zeta}+1}\left[\int_0^1 e^{-F_{\text{int}}(W)}\frac{\sin^2(\pi(\zeta - \bar{\zeta} - \bar{Z}(\zeta_0,\bar{\zeta},W)))}{\pi^2}dW\right]\sigma(\zeta)d\zeta = \upsilon(\tau). \tag{3.149}$$

This equation can be studied basically in the same manner as (3.131) with the only difference that there is an integration on the variable W, instead of just the particular choice of $\alpha = \frac{1}{2}$ that was made in the transition problem.

Equation (3.149) can be analyzed in a manner analogous to (3.131). Arguing as in the derivation of (3.139) we obtain:

$$\begin{aligned}
&\frac{d^3(\upsilon)}{d\bar{\zeta}^3} + 4\pi^2 \frac{d}{d\bar{\zeta}}(\upsilon) \\
&\quad = \frac{9\lambda''(\bar{\zeta}+1)}{(f'(\bar{\zeta}+1))^2} \int_0^1 \frac{e^{-F_{\mathrm{int}}(W)} dW}{(W-1)^2} - \frac{9\lambda'(\bar{\zeta}+1)}{f'(\bar{\zeta}+1)} \int_0^1 e^{-F_{\mathrm{int}}(W)} \Omega'(-F_{\mathrm{int}}(W)) dW \\
&\qquad + 9\lambda(\bar{\zeta}+1) \int_0^1 e^{-F_{\mathrm{int}}(W)} \Omega''(-F_{\mathrm{int}}(W)) dW \\
&\qquad - 9 \int_{\zeta_0}^{\bar{\zeta}+1} f'(\zeta) \left[\int_0^1 e^{-F_{\mathrm{int}}(W)} \Omega'''(f(\zeta) - f(\bar{\zeta}+1) - F_{\mathrm{int}}(W)) dW \right] \lambda(\zeta) d\zeta \\
&\qquad - 9 \int_{\zeta_0}^{\bar{\zeta}+\delta_1(\bar{\zeta})} f'(\zeta) \Psi''' \\
&\qquad \times \left(f(\zeta) - f\left(f^{-1}\left(f(\bar{\zeta}+1) + F_{\mathrm{int}}(W) - F_{\mathrm{int}}\left(\frac{1}{2}\right)\right) - 1\right)\right) \lambda(\zeta) d\zeta.
\end{aligned} \tag{3.150}$$

Equation (3.150) might be analysed exactly as (3.139) in Sect. 3.4.4. As in that case this equation has two different time scales. The shorter time scale is of order $\frac{1}{f'(\bar{\zeta}+1)}$ and is the scale associated to the terms containing derivatives on λ as well as the integral terms. The resulting operators yield stable behaviour as proved in Appendix 3. Then, in the longer time scale where $\bar{\zeta}$ varies quantities of order one we can approximate (3.150) as in Sect. 3.4.4 to obtain:

$$\lambda(\bar{\zeta}+1) - \lambda(\bar{\zeta}) = \frac{1}{2}\left[\frac{d^3(\upsilon)}{d\bar{\zeta}^3} + 4\pi^2 \frac{d}{d\bar{\zeta}}(\upsilon) \right] \tag{3.151}$$

and the desired stability of the self-similar behaviour follows.

4 Concluding Remarks

In this paper, a formalism that allows to approximate asymptotically the noncompactly supported solutions of the LSW that approach to self-similar solutions has been developed. An analogous, although fully rigorous analysis was made in [12, 13] for compactly supported initial data. As in the case of compactly supported initial data we have obtained that the resulting LSW dynamics might be approximated by means of integro-differential equations. There are, however, several crucial differences between both cases. In the compactly supported case, at a given time, the only particles that are relevant in the description of the long time asymptotics of the solutions are those who are vanishing in a time of order one. On the contrary, in the noncompactly supported case, there are two groups of particles playing a relevant role in the description of the dynamics of the LSW model near self-similar solutions. A first group is the set of particles whose remaining life-time is of order one, exactly as in the compactly supported case. On the other hand, the subset of the family of particles with a long life-time expectancy, whose radius is close to the so-called critical radius plays also a relevant role in the dynamics of the LSW system for noncompactly supported data. This fact does not have a natural correspondence in the compactly supported case. From the mathematical point of view, the main consequence is that, for compactly supported solutions, the LSW dynamics might be approximated by means of a set of integral equations of convolution type with fast decaying kernels. In the noncompactly supported case the final equations describing the evolution of the solutions are integro-differential equations having two time scales that can be studied using multiple scale methods.

The analysis in this paper suggests that there exists a large class of noncompactly supported initial data yielding self similar behaviours for long times. However, it has been also shown that not all the initial data yield such self-similar behaviour.

Acknowledgements I thank B. Niethammer for several illuminating discussions during the realization of this work, as well as an anonymous referee for very clarifying remarks. This work is partially supported by DGES Project MTM2004-05634.

Appendix 1: A Local Existence Theorem for the Transition Problem

In this Appendix we prove that the transition problem (3.53–3.57) is locally solvable in time.

Theorem 10 *Let us define* $W(s;W_0)$ *as the solution of the problem*:

$$W_s = -2 + 3W^{1/3} - W + 3\beta(s)W^{1/3}, \tag{5.1}$$

$$W(0;W_0) = W_0. \tag{5.2}$$

Then, for a given function $\omega(\cdot) \in C^1[0,\delta]$ *for some* $\delta > 0$ *and satisfying* $\omega(0) = \alpha$, $\alpha \in (0,1)$ *there exists an unique continuous function* $\beta(\cdot)$ *defined in* $s \in [0,T]$, *for some* $T > 0$ *such that*:

$$W(\tau;\omega(\tau)) = \alpha \tag{5.3}$$

for $\tau \in [0,T]$. *Moreover, suppose that for some* $\bar{W}_0 > 0$ *and* $\beta(s)$ *defined in* $s \in [0,\bar{\tau}]$ *the function* $W(s;\bar{W}_0)$ *is defined and it remains positive in the interval* $s \in [0,\bar{\tau}]$. *Then, for any function* $\omega(\cdot) \in C^1[\bar{\tau},\bar{\tau}+\delta]$ *for some* $\delta > 0$, *there exists* $\beta(\cdot)$ *defined in* $s \in [\bar{\tau},\bar{\tau}+T]$ *such that* (5.3) *holds for* $\tau \in [\bar{\tau},\bar{\tau}+T]$.

Proof The proof is just a fixed point argument. Integrating both sides of (5.1) with $W_0 = \omega(\tau)$ we obtain:

$$\begin{aligned} &W(\tau;\omega(\tau)) - \omega(\tau) \\ &\quad = \int_0^\tau \left[-2 + 3(W(s;\omega(\tau)))^{1/3} - W(s;\omega(\tau)) + 3\beta(s)(W(s;\omega(\tau)))^{1/3}\right] ds. \end{aligned}$$

Using (5.3) this equation becomes:

$$\alpha - \omega(\tau) = \int_0^\tau \left[-2 + 3(W(s;\omega(\tau)))^{1/3} - W(s;\omega(\tau)) + 3\beta(s)(W(s;\omega(\tau)))^{1/3}\right] ds. \tag{5.4}$$

Differentiating (5.4) and using again (5.3) we obtain:

$$-\omega'(\tau) = \big[-(2+\alpha) + 3(\alpha)^{1/3}\big] + 3(\alpha)^{1/3}\beta(\tau) + \omega'(\tau)\int_0^{\tau} \big[(W(s;\omega(\tau)))^{-2/3} - 1 + \beta(s)(W(s;\omega(\tau)))^{-2/3}\big]\frac{\partial W(s;\omega(\tau))}{\partial W_0}ds. \tag{5.5}$$

Differentiating (5.1), (5.2) with respect to W_0 and integrating the resulting differential equation we obtain:

$$\frac{\partial W(s;\omega(\tau))}{\partial W_0} = \exp\left(\int_0^{s} \big[(W(\xi;\omega(\tau)))^{-2/3} - 1 + \beta(\xi)(W(\xi;\omega(\tau)))^{-2/3}\big]d\xi\right)$$

and plugging this formula into (5.5) we obtain the equation:

$$\begin{aligned}&3\alpha^{1/3}\beta(\tau) + \omega'(\tau)\int_0^{\tau} \big[(W(s;\omega(\tau)))^{-2/3} - 1 + \beta(s)(W(s;\omega(\tau)))^{-2/3}\big]\\ &\quad\times e^{\int_0^s [(W(\xi;\omega(\tau)))^{-2/3}-1+\beta(\xi)(W(\xi;\omega(\tau)))^{-2/3}]d\xi}ds\\ &= -\omega'(\tau) + \big[(2+\alpha) - 3(\alpha)^{1/3}\big],\end{aligned} \tag{5.6}$$

where:

$$\begin{aligned}&W(s;\omega(\tau))\\ &\quad= \omega(\tau) + \int_0^{\tau} \big[-2 + 3(W(s;\omega(\tau)))^{1/3} - W(s;\omega(\tau)) + 3\beta(s)(W(s;\omega(\tau)))^{1/3}\big]ds.\end{aligned} \tag{5.7}$$

Equation (5.6), (5.7) can be solved using a contractive fixed point argument on the space $\beta(\cdot) \in C[0,T]$ with the uniform norm. □

Appendix 2: Some Properties of the Function $f(\zeta)$

In this Appendix we collect several properties of the function $f(\zeta)$ that have been used repeatedly in Sect. 3. The main result of this Appendix is the following:

Theorem 11 *Suppose that S satisfies (3.108). Given $f \in C^3[\zeta_0, \zeta_0+1]$ satisfying the compatibility conditions (3.80–3.83) we define a function $f \in C^3[\zeta_0, \zeta_0+1)$ by means of (3.77), (3.78). Then, f satisfies the following:*

$$f(\zeta) \gg \exp(\exp(\exp(\dots\exp(\zeta)))) \quad as\ \zeta \to \infty \tag{6.1}$$

for any finite number of iterated exponentials.

$$f^{(k-1)}(\zeta) \ll f^{(k)}(\zeta) \ll (f(\zeta))^{1+\varepsilon} \quad as\ \zeta \to \infty \tag{6.2}$$

for any $\epsilon > 0$, and any $k = 1, 2, 3$.

$$f(\zeta+\delta) \gg f(\zeta) \quad as\ \zeta \to \infty \tag{6.3}$$

for any $\delta > 0$.

$$f\left(\zeta + \frac{C}{f(\zeta)}\right) \sim f(\zeta) \quad as\ \zeta \to \infty \tag{6.4}$$

for any $C > 0$.

Proof Property (6.1) just follows iterating (3.77) more than k times. Property (6.3) can be proved in a similar manner. Indeed, iterating (3.77) by means of an exponential function it follows that,since $f(\zeta) \to \infty$, that $f(\zeta + \delta) - f(\zeta) \to \infty$, Then

$$\begin{aligned} f(\zeta + \delta) &= S^{-1}(f(\zeta + \delta - 1)) = S^{-1}([f(\zeta + \delta - 1) - f(\zeta - 1)] + f(\zeta - 1)) \\ &\gg S^{-1}(f(\zeta - 1)) = f(\zeta) \end{aligned}$$

as $\zeta \to \infty$.

Property (6.2) follows differentiating (3.77) that yields

$$f'(\zeta) = S'(f(\zeta + 1))f'(\zeta + 1) \sim \frac{a}{f(\zeta + 1)} f'(\zeta + 1). \tag{6.5}$$

Iterating (5) to estimate $f'(\zeta + 1)$ it follows from (6.1) that $f'(\zeta) \to \infty$. Combining this with (5) we obtain the first inequality in (6.2) with $k = 1$. On the other hand, combining (3.77) and (5) we obtain

$$\frac{f'(\zeta + 1)}{f(\zeta + 1)} = \left[\frac{1}{S'(f(\zeta + 1))} \frac{f(\zeta)}{S^{-1}(f(\zeta))}\right] \frac{f'(\zeta)}{f(\zeta)}. \tag{6.6}$$

The term between brackets is bounded by $Cf(\zeta)$. Iterating (6.6) we obtain

$$\frac{f'(\zeta_0 + n)}{f(\zeta_0 + n)} = \prod_{\ell=0}^{n-1}\left[\frac{1}{S'(f(\zeta_0 + 1 + \ell))} \frac{f(\zeta_0 + \ell)}{S^{-1}(f(\zeta_0 + \ell))}\right] \frac{f'(\zeta_0)}{f(\zeta_0)}. \tag{6.7}$$

The product in (6.7) can be bounded as

$$\prod_{\ell=0}^{n-1}[Cf(\zeta_0 + \ell)] \tag{6.8}$$

and taking the logarithm of this expression and taking into account that $\frac{f(\zeta+1)}{f(\zeta)} \geq 2$ for ζ large enough we obtain, after adding a geometric series, an upper estimate for the product in (6.8) of the form

$$\exp(B \log(f(\zeta_0 + n - 1))) = (f(\zeta_0 + n - 1))^B$$

for some $B > 0$, whence (6.7) yields

$$f'(\zeta) \leq Cf(\zeta)(f(\zeta - 1))^B \tag{6.9}$$

and since (3.77) implies that $f(\zeta - 1) \leq C \log(f(\zeta))$ we obtain (6.2) for $k = 1$. The proof of (6.2) for $k = 2, 3$ is similar.

In order to show (4) we iterate (3.77) to compute $f(\zeta + \frac{C}{f(\zeta)})$

$$f\left(\zeta + \frac{C}{f(\zeta)}\right) = S^{-1}\left(S^{-1}\left(\ldots S^{-1}\left(f\left(\zeta + \frac{C}{f(\zeta)} - n\right)\right)\right)\right),$$

where the number of iterations n is such that $\zeta + \frac{C}{f(\zeta)} - n \in [\zeta_0, \zeta_0 + 1]$. Since $f(\zeta)$ is huge we can approximate the terms $S^{-1}(f(\zeta + \frac{C}{f(\zeta)} - n))$ as $S^{-1}(f(\zeta -)n) + \frac{C(S^{-1})'(f(\zeta - n))}{f(\zeta)}$. Using this approximation in $n-1$ iterations, as well as (5) we obtain the approximation

$$f\left(\zeta + \frac{C}{f(\zeta)}\right) = S^{-1}\left(f(\zeta - 1) + \frac{Cf'(\zeta - 1)}{f(\zeta)}\right)$$

and using (6.9), (4) follows. Similar approximations might be derived for the derivatives. □

Appendix 3: Stability of Some Convolution Operators

In this Appendix we study the stability properties of the operators associated to times τ of order one in (3.139) and (3.150). To this end, we use the fact that the dominant operators in this time scale are convolution operators that can be studied using Laplace transforms.

7.1 Stability for the Transition Problem

Using the time variable $\tau = f(\bar{\zeta} + 1)$ we can transform (3.139), neglecting some small terms that are introduced in the error term as

$$\begin{aligned} &\frac{1}{(\alpha-1)^2}\frac{d^2\lambda(\tau)}{d\tau^2} - \Omega'(-F_{\text{int}}(\alpha))\frac{d\lambda(\tau)}{d\tau} + \Omega''(-F_{\text{int}}(\alpha))\lambda(\tau) \\ &\quad - \int_{\tau_0}^{\tau} \Omega'''(s - \tau - F_{\text{int}}(\alpha))\lambda(s)ds + H(\tau) \\ &\quad = O\left(\frac{1}{(f(\bar{\zeta}))^{\alpha}}\right) := r(\tau), \end{aligned} \tag{7.1}$$

where $H(\tau)$ contains the contribution of the term $\int_{\zeta_0}^{\bar{\zeta}+\delta_1(\bar{\zeta})} f'(\zeta)\Psi'''(f(\zeta) - f(\bar{\zeta}))\lambda(\zeta)d\zeta$ that might be approximated to the leading order as $C\lambda(S(\tau))$. Therefore, this term varies, for $\tau \to \infty$ in a longer time scale that the other terms in (7.1) and, as a consequence, if the convolution part of the operator can be shown to yield convergence, it would be possible to approximate (3.139) as (3.133). Taking the Laplace transform of (7.1) that we define as

$$\tilde{f}(z) = \int_0^{\infty} f(\tau)e^{-\tau z}d\tau$$

we obtain

$$\tilde{\lambda}(z) = \frac{\tilde{H}(z) + \tilde{r}(z) + C_1 z + C_2}{\frac{z^2}{(\alpha-1)^2} - \Omega'(-F_{\text{int}}(\alpha))z + \Omega''(-F_{\text{int}}(\alpha)) - \omega(z)},$$

where

$$\omega(z,\alpha) = \int_0^\infty \Omega'''(-\tau - F_{\text{int}}(\alpha))e^{-\tau z} d\tau.$$

Standard properties of the Laplace transform would show that the solutions of (7.1) approach to a constant value for $1 \ll \tau$ if the function

$$\Phi(z,\alpha) = \frac{z^2}{(\alpha-1)^2} - \Omega'(-F_{\text{int}}(\alpha))z + \Omega''(-F_{\text{int}}(\alpha)) - \omega(z,\alpha)$$

does not have zeroes in the half-plane $\{\mathrm{Re}(z) > 0\}$. (A similar argument, in a fully rigorous manner can be found in [12]). We have studied the zeroes of the function $\Phi(z)$ using the argument principle for the particular value $\alpha = \frac{1}{2}$. After some computations the function $\Phi(z; \frac{1}{2})$ becomes:

$$\Phi\left(z; \frac{1}{2}\right) = 4.0z^2 + 1.9024z + 0.23968 - 2e^{zF_{\text{int}}(\alpha)} I(z),$$

where

$$F_{\text{int}}(\alpha) = F_{\text{int}}\left(\frac{1}{2}\right) = 1.6622$$

and

$$I(z) = \int_{(\frac{1}{2})^{1/3}}^1 \frac{(-4 - 17Y + 12Y^6 - 29Y^3 - 45Y^2 + 60Y^5 + 68Y^4)}{(Y^2+Y+1)^5 Y^3} \times \exp\left(-z\left(\frac{4}{3}\log\left(1+\frac{Y}{2}\right) - \frac{Y}{(Y-1)} + \frac{5}{3}\log(1-Y)\right)\right) dY.$$

In Figs. 1, 2 we describe the image of a contour surrounding the half-plane $\{\mathrm{Re}(z) > 0\}$ for a large value of $R \to \infty$ and its image by means of this function. More precisely, Fig. 2 contains the image of the vertical segment $[0, 10i]$ by the function $\Phi(z; \frac{1}{2})$. The second picture provides a detail of the same curve near the origin. On the other hand Fig. 3 shows the image of the quarter of a circle $10e^{i\theta}$, $\theta \in [0, \frac{\pi}{2}]$ by means of $\Phi(z; \frac{1}{2})$. The second picture

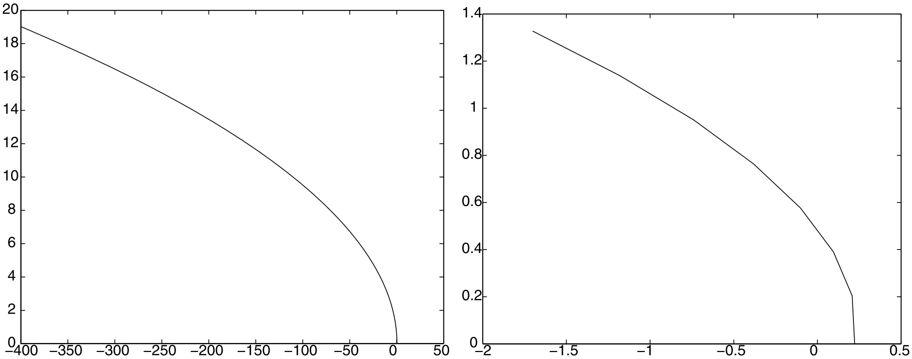

Fig. 2 The image by Φ of the positive imaginary axis

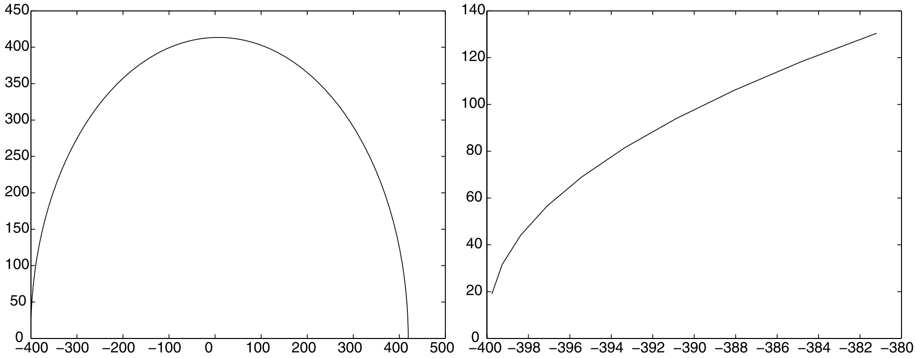

Fig. 3 The image by Φ of the portion of a big circle in the first quadrant

provides a detail of this curve for large values of $|z|$. The image by means of $\Phi(z;\frac{1}{2})$ of the segment $[-10i, 0]$ and the half-circle $10e^{i\theta}$, $\theta\in[-\frac{\pi}{2},0]$ can be obtained by reflection with respect to the real axis, since $(\Phi(z;\frac{1}{2}))^*=\Phi(z^*;\frac{1}{2})$. It follows from these pictures that the image by means of $\Phi(z;\frac{1}{2})$ of the curve $[-10i,10i]\cup\{z=10e^{i\theta}:\theta\in[\frac{\pi}{2},-\frac{\pi}{2}]\}$ does not surround the origin, and therefore, the argument principle implies that there are not zeroes of $\Phi(z;\frac{1}{2})$ in the half-plane $\{\mathrm{Re}(z)\geq 0\}$. This yields the desired stability.

7.2 Stability for the Linearized LSW Model

In the case of the linearization of the whole LSW model the study can be made in a similar manner. Using the time variable τ, (3.150) becomes, neglecting the smallest terms

$$\begin{aligned}&\left(\int_0^1\frac{e^{-F_{\mathrm{int}}(W)}dW}{(W-1)^2}\right)\frac{d^2\lambda(\tau)}{d\tau^2}-\left(\int_0^1 e^{-F_{\mathrm{int}}(W)}\Omega'(-F_{\mathrm{int}}(W))dW\right)\frac{d\lambda(\tau)}{d\tau}\\&\quad+\left(\int_0^1 e^{-F_{\mathrm{int}}(W)}\Omega''(-F_{\mathrm{int}}(W))dW\right)\lambda(\tau)\\&\quad-\int_{\tau_0}^{\tau}\left[\int_0^1 e^{-F_{\mathrm{int}}(W)}\Omega'''(s-\tau-F_{\mathrm{int}}(W))dW\right]\lambda(s)ds+H(\tau)+r(\tau),\end{aligned}\tag{7.2}$$

where as in (7.1), $r(\tau)$ converges to zero and $H(\tau)$ varies slowly in the time scale $S(\tau)$.

Equation (7.2) can be studied similarly to (7.1) using Laplace transforms. The stability properties of the convolution operator (including the derivatives) reduces then to prove that the function

$$\Psi(z)=\int_0^1\Phi(z,W)e^{-F_{\mathrm{int}}(W)}dW$$

does not have zeroes in the half-plane $\{\mathrm{Re}(z)>0\}$. This property can be shown exactly as in the previous Subsection using the argument principle. Figure 4 shows the image of the segment $[0,10i]$. The second picture provides a detail of the region close to the origin. Figure 5 shows the image by Ψ of the quarter of a circle $10e^{i\theta}$, $\theta\in[0,\frac{\pi}{2}]$. The second picture gives a detail of the region or large values of $|z|$.

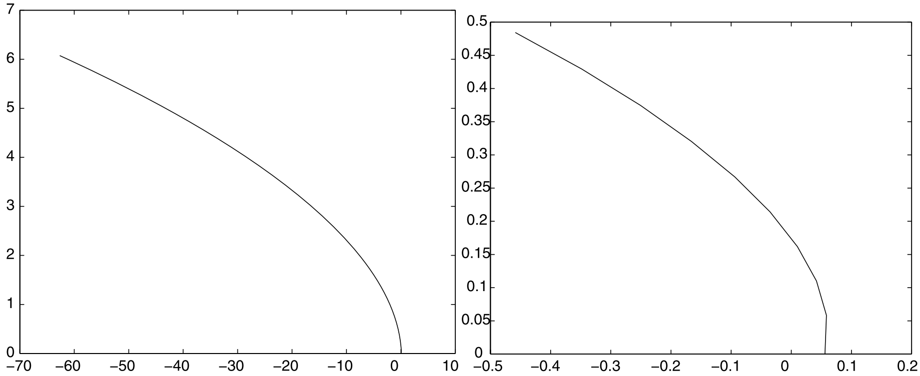

Fig. 4 The image by Ψ of the positive imaginary axis

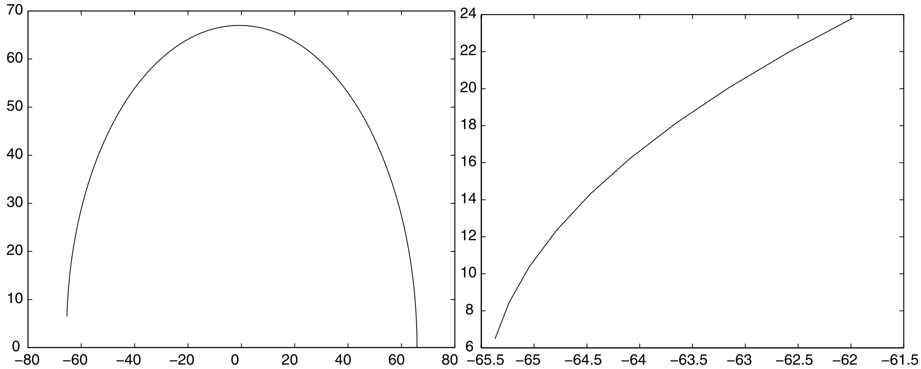

Fig. 5 The image by Ψ of the portion of a big circle in the first quadrant

References

1. Carr, J., Penrose, O.: Asymptotic behaviour in a simplified Lifshitz-Slyozov equation. Physica D **124**, 166–176 (1998)
2. Carrillo, J., Goudon, T.: A numerical study on large-time asymptotics of the Lifshitz-Slyozov system. J. Sci. Comput. **20**(1), 69–113 (2004)
3. Garroni, A., Niethammer, B.: Correctors and error-estimates in the homogenization of a Mullins-Sekerka problem. Ann. Inst. H. Poincaré Anal. Nonlineaire **194**, 371–393 (2002)
4. Giron, B., Meerson, B., Sasorov, P.V.: Weak selection and stability of localized distributions in Ostwald ripening. Phys. Rev. E **58**, 4213–6 (1998)
5. Hille, E.: Ordinary Differential Equations in the Complex Domain. Dover, New York (1997)
6. Lifshitz, I.M., Slyozov, V.V.: The kinetics of precipitation from supersaturated solid solutions. J. Phys. Chem. Solids **19**, 35–50 (1961)
7. Niethammer, B.: Derivation of the LSW theory for Ostwald ripening by homogenization methods. Arch. Ration. Mech. Anal. **147**(2), 119–178 (1999)
8. Niethammer, B., Otto, F.: Ostwald ripening: the screening length revisited. Calc. Var. **13**(1), 33–68 (2001)
9. Niethammer, B., Pego, R.L.: Non-self-similar behaviour in the LSW theory of Ostwald ripening. J. Stat. Phys. **95**(5/6), 867–902 (1999)
10. Niethammer, B., Pego, R.L.: On the initial value problem in the Lifshitz-Slyozov-Wagner theory of Ostwald ripening. SIAM J. Math. Anal. **31**(3), 457–485 (2000)
11. Niethammer, B., Velázquez, J.J.L.: Screening in interacting particle systems. Arch. Ration. Mech. Anal. **180**(3), 493–506 (2006)

12. Niethammer, B., Velázquez, J.J.L.: On the convergence to the smooth self-similar solution in the LSW model. Indiana Univ. Math. J. **55**, 761–794 (2006)
13. Niethammer, B., Velázquez, J.J.L.: Global stability and bounds for coarsening rates within the LSW mean-field theory for domain coarsening. Commun. Partial Differ. Equ. **31**, 1–30 (2006)
14. Velázquez, J.J.L.: The Becker-Döring equations and the Lifshitz-Slyozov theory of coarsening. J. Stat. Phys. **92**(1/2), 195–236 (1998)
15. Velázquez, J.J.L.: On the dynamics of the characteristic curves for the LSW model. Electron. J. Differ. Equ. **2006**(52), 1–51 (2006)
16. Wagner, C.: Theorie der Alterung von Niederschlägen durch Umlösen. Z. Elektrochem. **65**, 581–594 (1961)